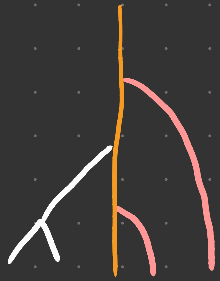
Identifying invertibility of bimodule categories

w/ Laurens Lootens & Frank Verstraete

arXiv: 2211.01947



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Q] Given $m \times n$ matrix $/\mathbb{C}$

$$X$$

is there Y such that:

$$XY = \mathbb{1} = YX$$

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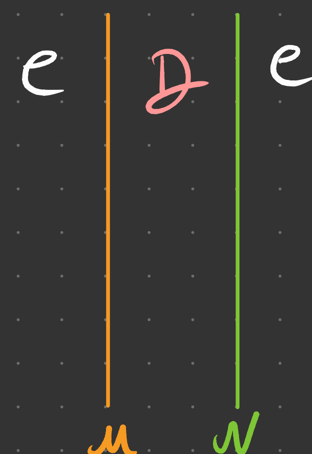
Q] Given

is there $\mathcal{N}$ such that:



$\cong \mathcal{C}$

A] ?

Overview

- Rules of the game

- Fusion cats & their modules
    - What are they here?

- Weak Hopf algebras & representations.

$\longleftrightarrow$ Simple formula characterizing invertibility

Theorem 1 (Invertibility)**.** *Let $\mathcal{C}$, $\mathcal{D}$ be unitary, fusion categories, and ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ an indecomposable, unitary, finitely semisimple, skeletal bimodule category. Then $\mathcal{M}$ is invertible as a $(\mathcal{C},\mathcal{D})$-bimodule category if and only if*

$$\operatorname{FPdim}\mathcal{C} = \operatorname{FPdim}\mathcal{D} \qquad \text{and}$$

$$\frac{1}{\operatorname{rk}\mathcal{M}} \sum_{\substack{a\in\operatorname{Irr}\mathcal{C}\\ b,d\in\operatorname{Irr}\mathcal{M}\\ \alpha,\beta,\mu,\nu}} \frac{d_a}{d_b^2}\, {}^{\mu}_{b}_{\alpha}\!\left[{}^{\bowtie}F^{\,d}_{abc}\right]^{\beta}_{d}{}_{\mu}\; {}^{\nu}_{b}_{\alpha}\!\left[{}_{\bowtie}F^{\,d}_{abc'}\right]^{\beta}_{d}{}_{\nu} = \delta_c^{c'},$$

for all c, $c' \in \operatorname{Irr}\mathcal{D}$.

- Associate algebra A to $\mathcal{C} \curvearrowright \mathcal{M}$
- Compute $\operatorname{Rep}(A)$ (this is Morita dual $\mathcal{C}^*_{\mathcal{M}}$)
- Is $\operatorname{Rep}(A) \cong \mathcal{D}$?

Can we check without computing $\operatorname{Rep}(A)$?

— Schur character orthogonality for A —

Game

Given a bimodule category, specified by its skeletal data,

determine whether it's invertible

$\mathcal{C} \curvearrowright \mathcal{M} \curvearrowleft \mathcal{D}$

$\mathcal{M} \otimes_{\mathcal{D}} \mathcal{M}^{op} \simeq \mathcal{C}$

Fusion-, Module-, Bimodule-categories

$\otimes$ unit

1) Finite set of simples $\operatorname{irr} \mathcal{C} = \{ 1, a, b, \dots \}$

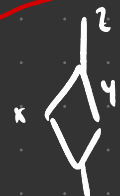
2) 
$\longleftrightarrow \quad \alpha \in \mathcal{C}(a \otimes b, c)$

3) 
$= \sum {}^{\otimes}F$


unitary matrix encoding associators

$(a \otimes b) \otimes c \simeq a \otimes (b \otimes c)$


$a = d_a$


$= \sqrt{\frac{d_x d_y}{d_z}} \; \Big|_z$

Example: Vec G

1) $\operatorname{irr} \mathrm{Vec}\, G = G$ as a set

2) $g \otimes h = gh$

3)

$${}_{gh}\left[{}^{\otimes}F^{ghk}_{g,h,k} \right]_{hk} = +1$$

Example: Rep G

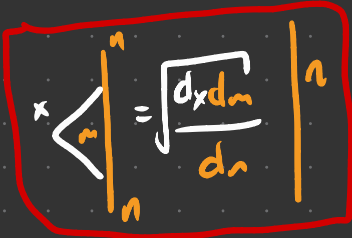
1) irr Rep G = irreducible reps of G

2) $\rho_x \otimes \rho_y \simeq \bigoplus_z N_{xy}^{z}\, \rho_z$

3)

$= \sum {}^{\beta}_{p\,\alpha} \left[F^{w}_{xyz}\right]^{\nu}_{q\,\mu}$

6j-symbols



Fusion-, Module-, Bimodule-categories

Given $\mathcal{C}$ fusion

$\mathcal{C} \triangleright \mathcal{M}$

1) Finite set of simples $\operatorname{irr}\mathcal{M} = \{m, n, \dots\}$

2) 

$\longleftrightarrow \quad \alpha \in \mathcal{M}(a \triangleright m, n)$

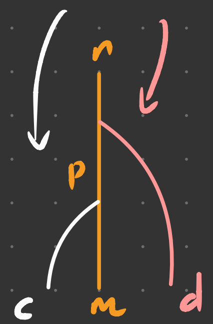
3)  $= \sum F^{\triangleright}$ 

unitary matrix encoding associators

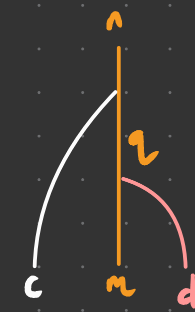
$(a \otimes b) \triangleright m \simeq a \otimes (b \triangleright m)$

Fusion-, Module-, Bimodule-categories

Given $\mathcal{C}, \mathcal{D}$ fusion, $\quad \mathcal{C} \curvearrowright \mathcal{M} \curvearrowleft \mathcal{D}$



$$= \sum {}_{p}\left[{}^{\bowtie}F^{n}_{cmd} \right]_{q}$$



<u>Skeletal data:</u> $\operatorname{irr}\mathcal{C}$, $\operatorname{irr}\mathcal{M}$, $\operatorname{irr}\mathcal{D}$

$$\left\{ {}^{\otimes}F, \ {}^{\triangleright}F, \ {}^{\bowtie}F, \ {}^{\triangleleft}F, \ {}^{\otimes}F \right\}$$

This is what we're given

Example. $\mathrm{Vec}\,G \curvearrowright \mathrm{Vec} \curvearrowleft \mathrm{Rep}\,G$

1 simple $*$ = 1-dim vector space.

Left

$g \triangleright * = *$

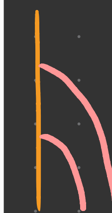
$=$


$= \sum \rho_x(g)_{ij}$


can check pentagon eqns

Right

$* \triangleleft x = \dim V_x \; *$


$= \sum CG$


$\mathrm{Vec}\,G \curvearrowright \mathrm{Vec} \circlearrowleft \mathcal{D}$

Property of the mixed associator:

$$\boxed{\frac{1}{|G|}\sum_g \chi_x(g)\,\chi_y(g^{-1}) = \delta_x^y \quad \text{for } x, y \text{ irred.}}$$

Schur's 1st orthogonality relation.

If we had chosen a different $\mathcal{D}$, this wouldn't work.

* Reducible reps
* Multiple copies
* Missing irreps

Can check $\mathcal{D} \cong \mathrm{Rep}\, G$

How to generalize?

Morita Dual

Given $\mathcal{C} \curvearrowright \mathcal{M}$, can construct a <u>unique</u> FC $\mathcal{C}^*_{\mathcal{M}}$, the dual, such that

$$\mathcal{C} \curvearrowright \mathcal{M} \curvearrowleft \mathcal{C}^*_{\mathcal{M}}$$

is invertible.

$$\mathcal{C}^*_{\mathcal{M}} \cong \mathrm{End}_{\mathcal{C}}(\mathcal{M})$$

$$m \triangleleft F := F(m)$$

part of the data of module functor

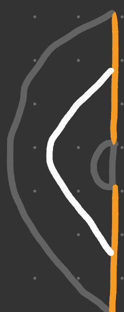
$$(a \triangleright m) \triangleleft F = F(a \triangleright m) \cong a \triangleright F(m) = a \triangleright (m \triangleleft F)$$

Constructing e^*_μ

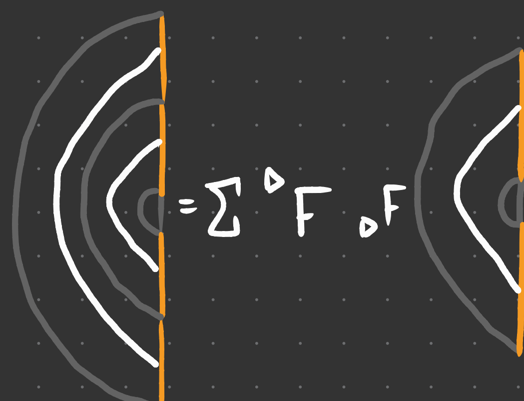
Fact: $e^*_\mu \cong \mathrm{Rep}\big(\underbrace{\mathrm{Ann}_e(\mu)}_{\text{Algebra}}\big)$

Computing this is topic of earlier papers SEE WEBSITE

Basis



Product

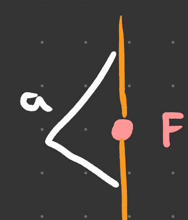


Unit

Roughly: to specify a module functor, we need vector spaces $\mathcal{M}(F(m), n)$

For this we use the v.s. underlying rep.

Natural isomorphisms: $F(a \triangleright m) \simeq a \triangleright F(m)$ provided by action of the algebra.



$\mathrm{Vec}\, G \curvearrowright \mathrm{Vec}$:

$T_g :=$ 

$T_g \cdot T_h =$ 

$= T_{gh}$

$$\mathrm{Rep}(\mathrm{Ann}) \cong \mathrm{Rep}(\mathbb{C}G)$$

Recall: We want to show $\mathcal{C} \curvearrowright \mathcal{M} \curvearrowleft \mathcal{D}$ is invertible.

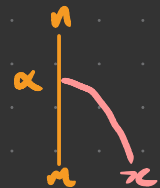
Reduces to showing that

$$\mathcal{D} \cong \mathrm{Rep}(Am_e(\mathcal{M}))$$

Representations of $\mathrm{Ame}(\mathcal{M})$ from $\mathcal{D}$

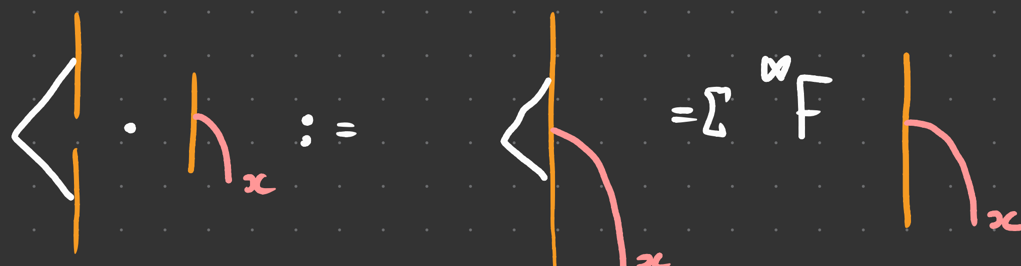
Pick simple $x \in \mathcal{D}$

Define vector space V_x with basis

$\Big\{$ 

$\Big| \; m, n \in \operatorname{irr} \mathcal{M} \, ; \, \alpha \leq \dim \mathcal{M}(m \triangleleft x, n) \Big\}$

Action of $\mathrm{Ame}(\mathcal{M})$:



Example: $\mathrm{Vec}_{\mathbb{Z}_2} \circlearrowright \mathrm{Vec} \circlearrowleft \mathrm{Rep}\, S_3$

1  $= \rho_\pi(1)_{00}$  $+ \rho_\pi(1)_{01}$ 

$$\rho_\pi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_\pi(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

π restricts to $1 \oplus \sigma$ on $\mathbb{Z}_2$ subgroup

$$(\chi_\pi, \chi_\pi) > 1$$

— Want to show $\mathcal{D} \simeq \mathcal{C}_{\mu}^{*} \simeq \mathrm{Rep}(\mathrm{Ann}_e(\mu))$ —

* Check simple objects label <u>distinct</u> <u>irreducible</u> representations. Character orthogonality?

$\chi_x\Big($  $\Big) = \mathrm{Tr}\, \rho_x\Big($  $\Big) = \sum \left({}^{\bowtie}F\right)$

* Check we haven't missed any. Dimension condition

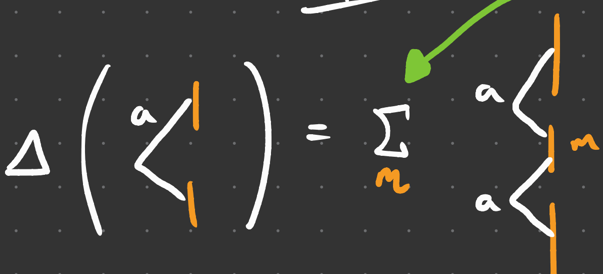
Can we use extra structure to construct an inner product so that $(\chi_i, \chi_j) = \delta_i^j$ for irreducible characters?

* Fact: $\mathrm{Ann}_{\mathcal{C}}(\mathcal{M})$ is a C^*-weak Hopf algebra
(pure)

— WHA: $\underbrace{\text{Algebra} + \text{Coalgebra}}_{\text{weakened compatibility}} + \text{Antipode}$

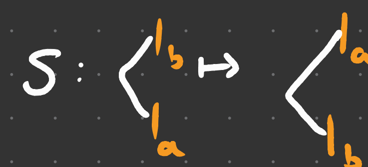
$\Delta 1 \neq 1 \otimes 1$.

coproduct:



Counit

= dimensions



Haar Integrals in WHA

Vector Λ in A such that

$$x\Lambda = \varepsilon(1_{(1)}x)\,1_{(2)}\Lambda \qquad \forall x \in A$$

$$\Lambda x = \Lambda 1_{(1)}\,\varepsilon(x 1_{(2)})$$

+ normalization condition.

Hopf : $x\Lambda = \Lambda x = \varepsilon(x)\Lambda$

Generalizes $\dfrac{1}{|G|}\sum\limits_{g\in G} g$ in the case $A = \mathbb{C}G$

Always exists in case A is C^*

Claim: $(\chi_x, \chi_y) := \langle \chi_x \chi_y^*, \Lambda \rangle = \delta_i^j$

Irreducible characters of WHA

$\hat{A} := \mathrm{Hom}(A, \mathbb{C})$ also a WHA

G Boehm
D Nikshych
V Ostrik
⋮

$$\chi_x \chi_y^* = \sum N_{x\bar{y}}^z \chi_z$$

so $(\chi_x, \chi_y) = \sum N_{x\bar{y}}^z \chi_z(\Lambda)$

[Boehm 99] $\chi_z(\Lambda) = \delta_z^{\text{trivial}}$

Pf: Trivial rep$^{\underline{n}}$ is image of $\varepsilon(1_{(1)} -)1_{(2)}$

Look for Homs.

Final result:

$$(\chi_x, \chi_y) = \delta_x^y \quad \text{iff} \quad x = y \quad \text{are irreps.}$$

Another way to evaluate:

$$(\chi_x, \chi_y) = \langle \chi_x \chi_y^*, \Lambda \rangle = \chi_x(\Lambda_{(1)}) \overline{\chi_y(S(\Lambda_{(2)})^*)}$$

Plug in $\chi_x\Big($  $\Big) = \Sigma({}^{\bowtie}F)$

If $\mathcal{D} \cong e_\mu^*$, then

$$\frac{1}{\operatorname{rk}\mathcal{M}} \sum_{\substack{a\in\operatorname{Irr}\mathcal{C}\\ b,d\in\operatorname{Irr}\mathcal{M}\\ \alpha,\beta,\mu,\nu}} \frac{d_a}{d_b^2}\, {}^{\mu}_{b}{}_{\alpha}\!\left[{}^{\bowtie}F^{\,d}_{abc}\right]^{\beta}_{d\,\mu}\; {}^{\nu}_{b}{}_{\alpha}\!\left[{}^{\bowtie}F^{\,d}_{abc'}\right]^{\beta}_{d\,\nu} = \delta_c^{c'}$$

$\forall\, c, c' \in \mathcal{D}$

otherwise : 1) $\mathcal{D}$ Missing some irreps
$\Rightarrow$ Dimensions won't match.

2) $x \in \mathcal{D}$ reducible $\Rightarrow (\chi_x, \chi_x) > 1$

3) x, y label same rep $\Rightarrow (\chi_x, \chi_y) \neq 0$.

Orthogonality of characters for C^*-WHA gives:

Theorem 1 (Invertibility). *Let $\mathcal{C}$, $\mathcal{D}$ be unitary, fusion categories, and ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ an indecomposable, unitary, finitely semisimple, skeletal bimodule category. Then $\mathcal{M}$ is invertible as a $(\mathcal{C},\mathcal{D})$-bimodule category if and only if*

$$\operatorname{FPdim}\mathcal{C} = \operatorname{FPdim}\mathcal{D} \qquad \text{and} \tag{19a}$$

$$\frac{1}{\operatorname{rk}\mathcal{M}} \sum_{\substack{a\in\operatorname{Irr}\mathcal{C}\\ b,d\in\operatorname{Irr}\mathcal{M}\\ \alpha,\beta,\mu,\nu}} \frac{d_a}{d_b^2}\, {}^{\mu}_{b}{}_{\alpha}\!\left[{}^{\bowtie}F^{\,d}_{abc}\right]^{\beta}_{d}{}_{\mu}\; {}^{\nu}_{b}{}_{\alpha}\!\left[{}_{\bowtie}F^{\,d}_{abc'}\right]^{\beta}_{d}{}_{\nu} = \delta_c^{c'}, \tag{19b}$$

for all c, $c' \in \operatorname{Irr}\mathcal{D}$.

Can also extend to matrix element orthog:

$$\sum_g \rho_x(g)_{\alpha\beta}\, \rho_y(g^{-1})_{\beta'\alpha'} = \frac{|G|}{\dim V_x}\, \delta_x^y\, \delta_\alpha^{\alpha'}\, \delta_\beta^{\beta'}$$

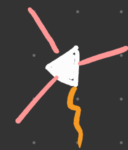
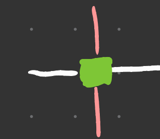
Schur's 2nd orthogonality relation.

Theorem 2 (Orthogonality of matrix elements)**.** *Let $\mathcal{C}$ be a unitary fusion category, and ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}^*_{\mathcal{M}}}$ an indecomposable, unitary, finitely semisimple, invertible bimodule category.*

Let c, c' be simple objects in $\mathcal{C}^*_{\mathcal{M}}$, then

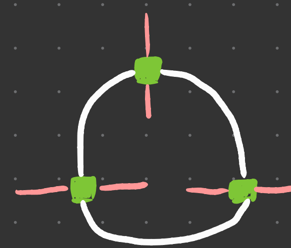
$$\sum_{\substack{a\\ \alpha,\nu}} d_a \; {}^{\beta}_{\substack{e\\ \alpha}}\!\left[{}^{\bowtie}F^{\,d}_{abc}\right]^{\nu}_{\substack{f\\ \mu}} \; {}^{\beta'}_{\substack{e\\ \alpha}}\!\left[{}_{\bowtie}F^{\,d}_{abc'}\right]^{\nu}_{\substack{f\\ \mu'}} = \delta^{c'}_{c}\,\delta^{\beta'}_{\beta}\,\delta^{\mu'}_{\mu}\,\frac{d_e d_f}{d_c} \tag{20}$$

Application: MPO-injectivity

PEPS$(\mathcal{M}, \mathcal{D})$ =  , MPO$(\mathcal{C}, \mathcal{M}, \mathcal{D})$ = 

$\downarrow$ STATE( , M) $\qquad\qquad\qquad\qquad \downarrow$

Physical state Ψ_M (Not unique) $\qquad$ Quantum symmetries of Ψ_M

 =  $\Longrightarrow$ $\# \Psi \leq \mathrm{rk}\, Z(\mathcal{C})$

$(M = T^2)$

Theorem 2 (Orthogonality of matrix elements). *Let $\mathcal{C}$ be a unitary fusion category, and ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}^*_{\mathcal{M}}}$ an indecomposable, unitary, finitely semisimple, invertible bimodule category.*

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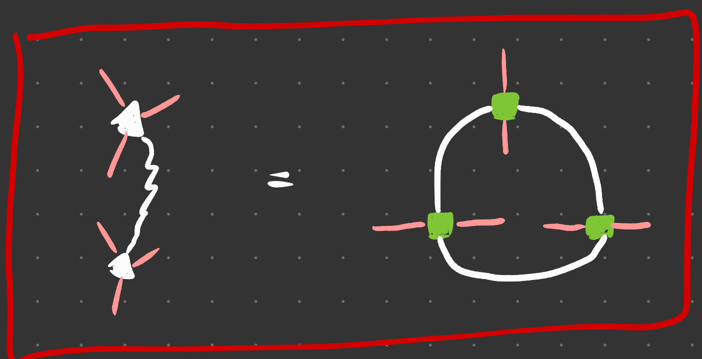
$$\sum_{\substack{a\\ \alpha,\nu}} d_a \begin{smallmatrix}\beta\\ e\\ \alpha\end{smallmatrix}\!\left[{}^{\bowtie}F^{\,d}_{abc}\right]^{\begin{smallmatrix}\nu\\ f\\ \mu\end{smallmatrix}} \begin{smallmatrix}\beta'\\ e\\ \alpha\end{smallmatrix}\!\left[\bowtie F^{\,d}_{abc'}\right]^{\begin{smallmatrix}\nu\\ f\\ \mu'\end{smallmatrix}} = \delta_{c}^{c'}\,\delta_{\beta}^{\beta'}\,\delta_{\mu}^{\mu'}\,\frac{d_e d_f}{d_c} \tag{20}$$

SAME EQN



Outlook

- Extending classical results to quantum symmetries
  - Schur orthogonality relations
  - Wigner-Eckart thm : constraints on symmetric tensors

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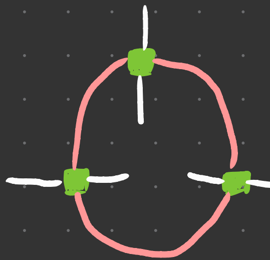
- Beyond finite case?
  - Fusion n-categories? weak Hopf + ?

Plugging skeletal data into  = 

Yields eq.^ 20 so $\boxed{\text{MPO-injectivity} = \text{invertible bimodule}}$

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Questions ?

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