

# Defects in topological phases

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w/ D. Barter & C. Jones

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w/ D. Barter

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Topological phase =  $(2+1)\mathcal{D}$  string-net model

Defect = things we can add to the bare model

Boundaries, excitations, domain walls, . . .

Several ways of describing/defining LW models

String types

Excitations

Lattice Hamiltonian

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$e$



$Z(e)$

Fusion category

Modular category

Drawing silly pictures

$$|gs\rangle := | \rangle + \alpha | \text{blue loop} \rangle + \beta | \text{green loop} \rangle + \gamma | \text{blue loop with green loop} \rangle + \dots$$

Declare this is a ground state.

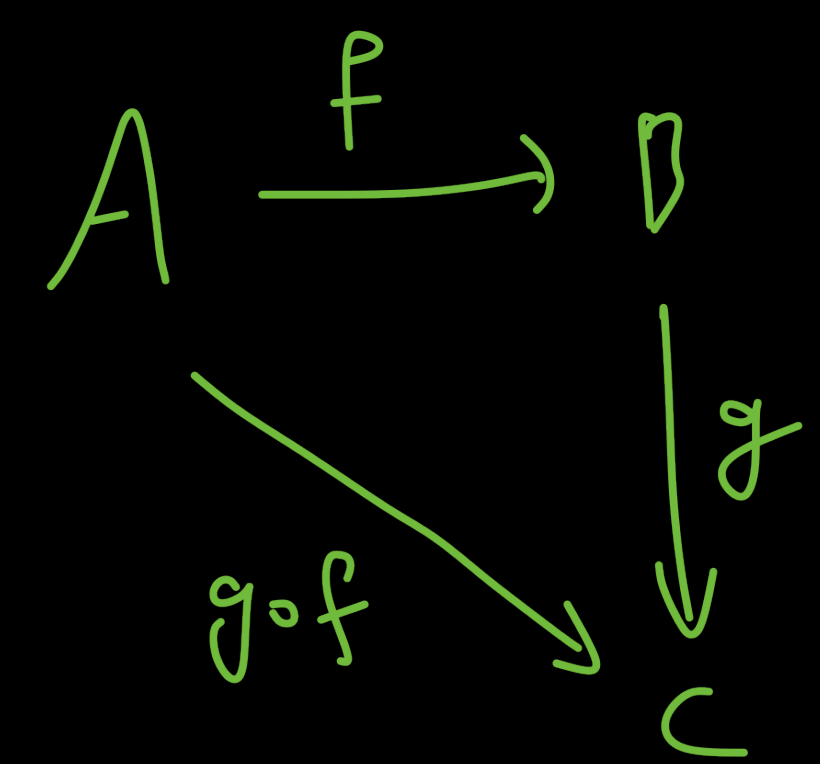
Topological : Pictures related by "local moves"  $\Rightarrow$  occur in superposition

Consistency  $\Rightarrow$  Fusion category.

# Brief intro to categories\*

set of objects  
 $\{A, B, C, \dots\}$

For each pair  $A, B \in \mathcal{C}$   
a set of morphisms,  $\text{Hom}(A, B)$   
 $f: A \rightarrow B$



\* Skipping many important details

Example

Any set  $S$  as a category:

$|S|$  objects       $\text{ob } \mathcal{C} = S$

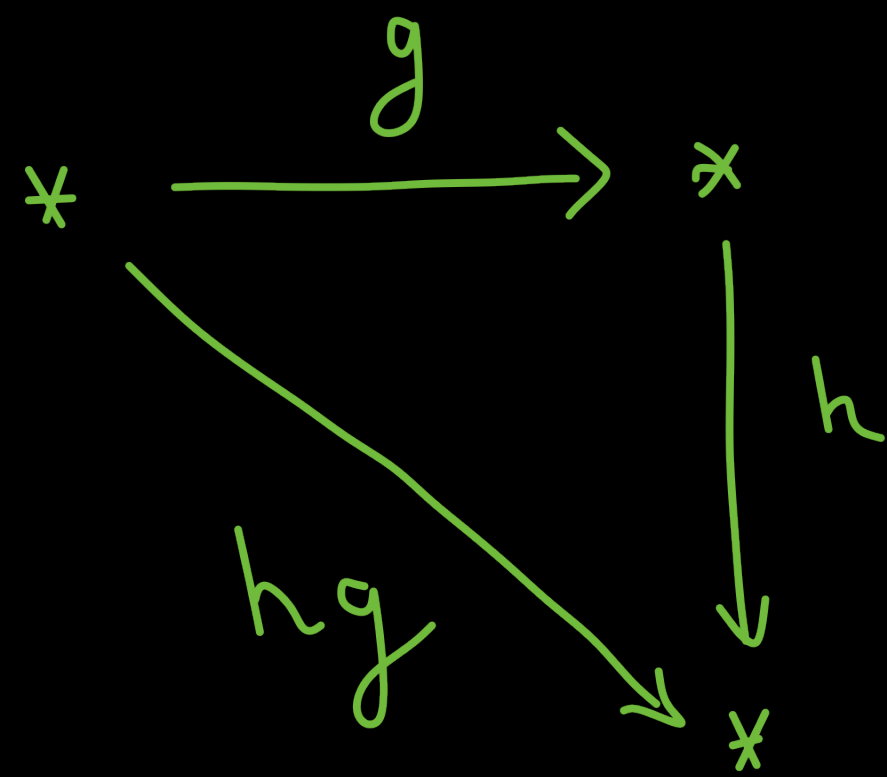
$$\text{Hom}(s, s') = \begin{cases} \text{identity} & \text{if } s = s' \\ \emptyset & s \neq s' \end{cases}$$

Example

Finite group  $G$  as a category:

1 object :  $*$

$$\text{Hom}(*, *) = G$$



Example :  $\text{Vec}$  : Category of  $\mathbb{C}$ -vector spaces

objects = {finite dimensional vector spaces}

$\text{Hom}(V, W) = \{\text{linear maps } V \rightarrow W\}$ .

simple objects : 1D vector space  $\mathbb{C}$  (up to isomorphism)

All other objects can be written as

$$\mathbb{C} \oplus \mathbb{C} \oplus \dots \oplus \mathbb{C}$$



Example

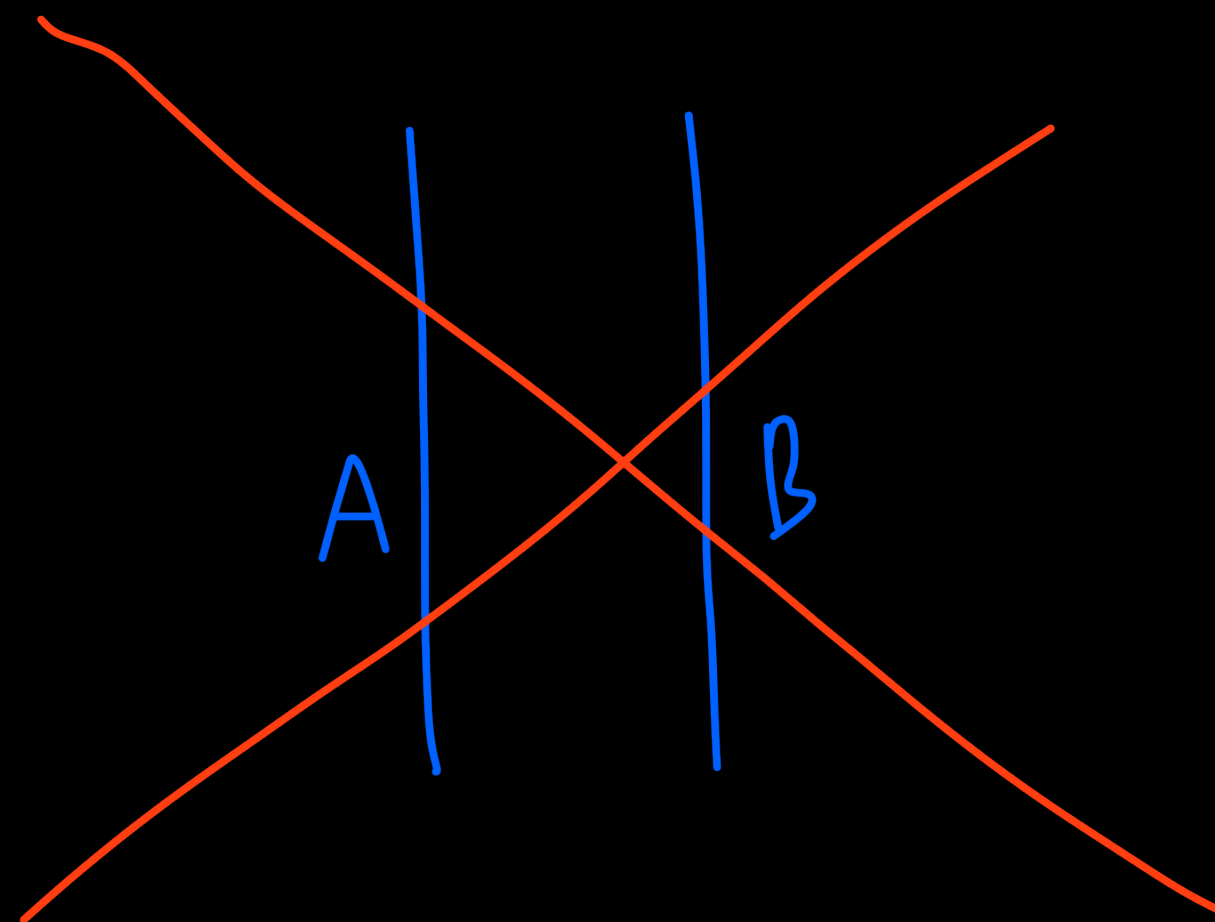
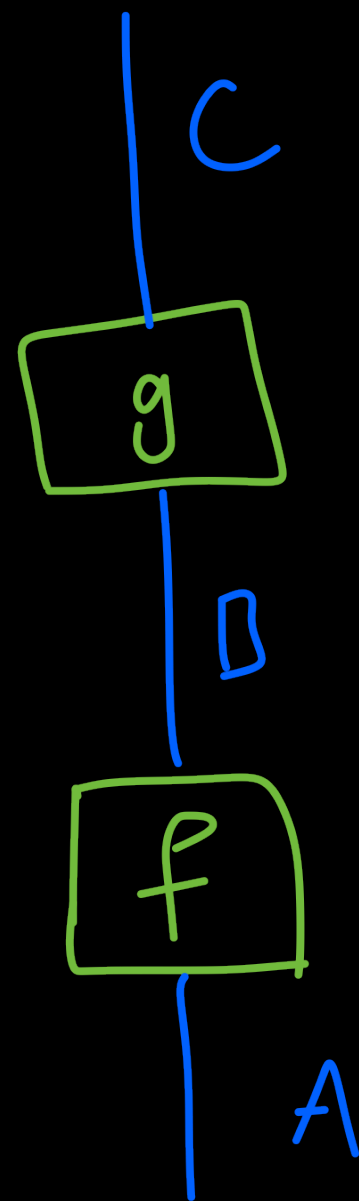
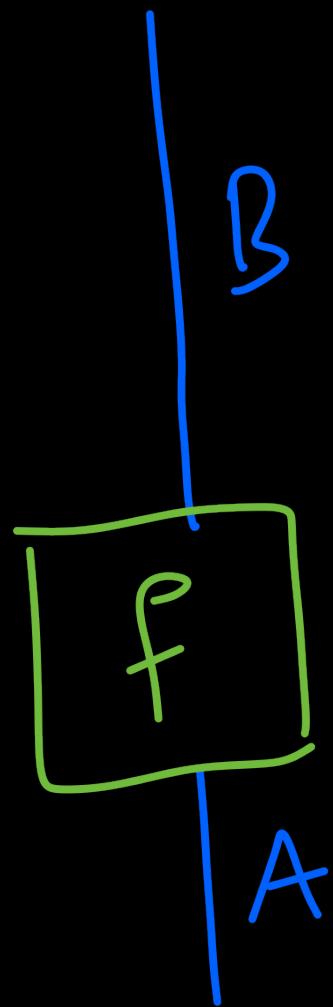
$\text{Vec}(G)$  :  $G$  graded vector spaces

objects :  $\left\{ \text{pairs } \left( V \text{ a vector space, } V = \bigoplus_{g} V_g \right) \right\}.$

simple objects :  $(\mathbb{C}, g)$

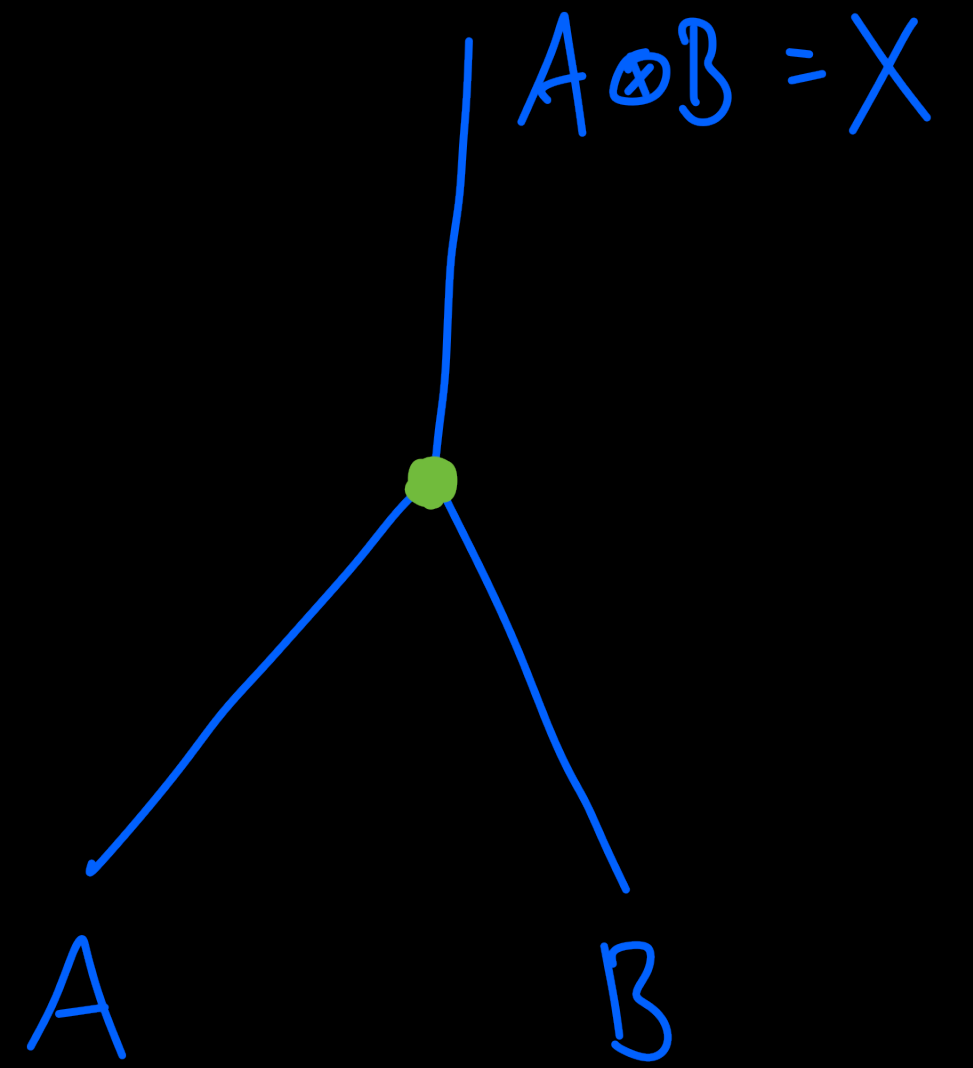
morphisms : Linear maps  $V \rightarrow W$   
 $f(V_g) \subset W_g$

# Back to pictures

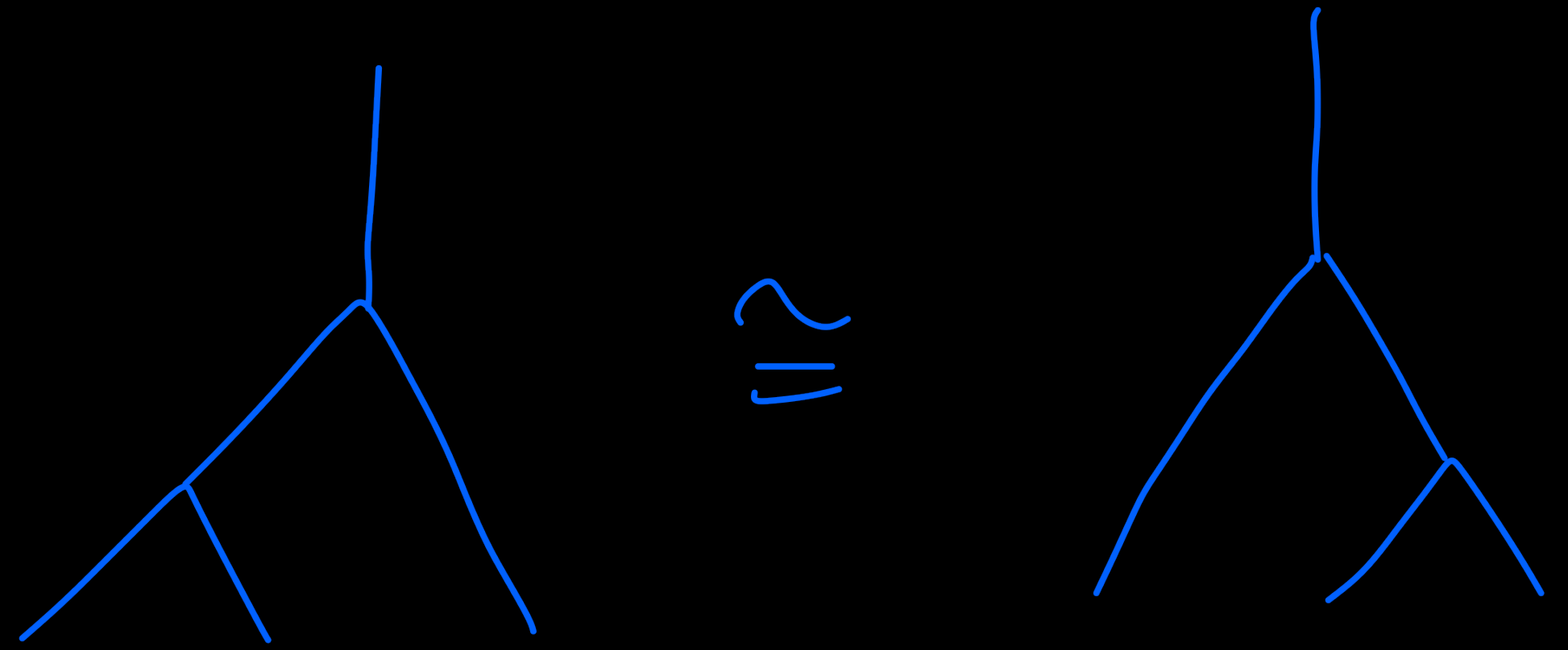


Fusion categories

$$\otimes: e \times e \longrightarrow e$$



$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$



Example

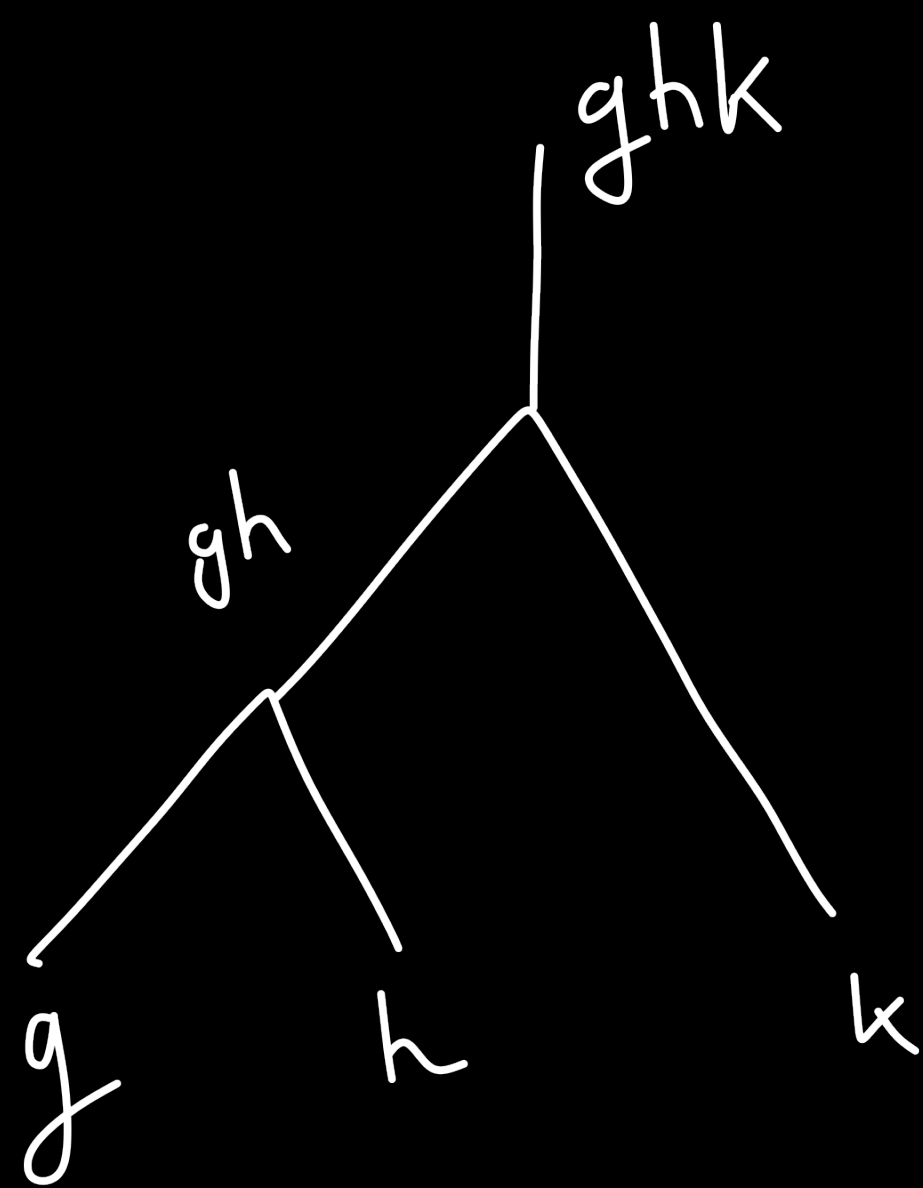
Vec as a FC:

$$\mathbb{C}^m \otimes \mathbb{C}^n \rightarrow \mathbb{C}^{mn}$$

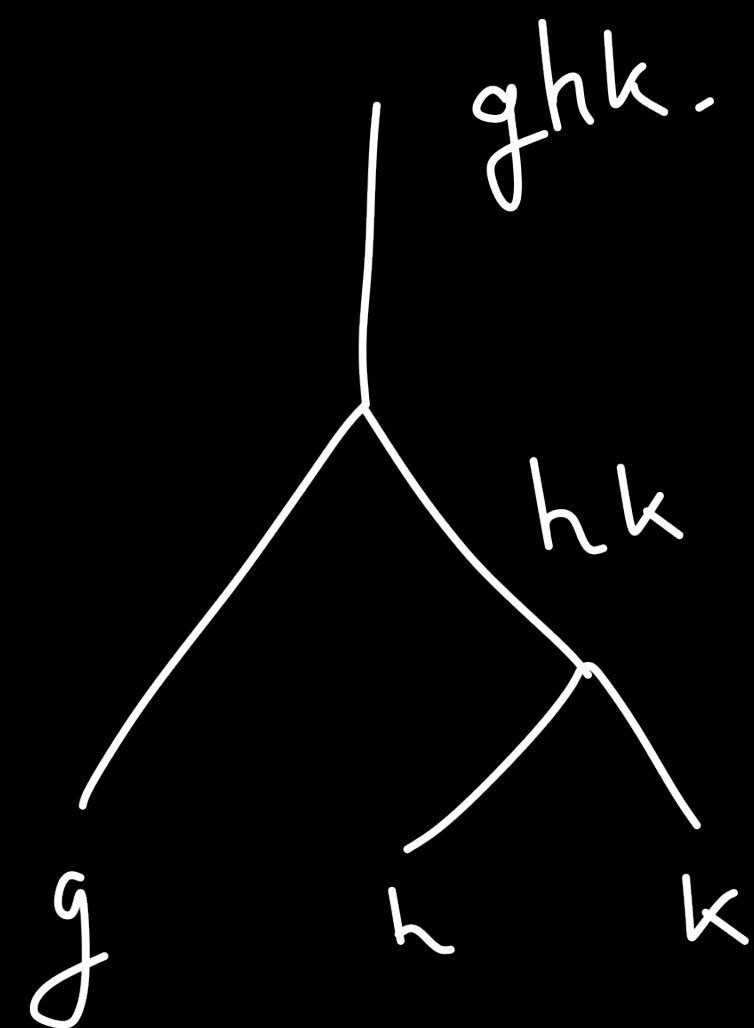
Example

$\text{Vec}(G)$

$$(\mathbb{C}, g) \otimes (\mathbb{C}, h) = (\mathbb{C}, gh)$$



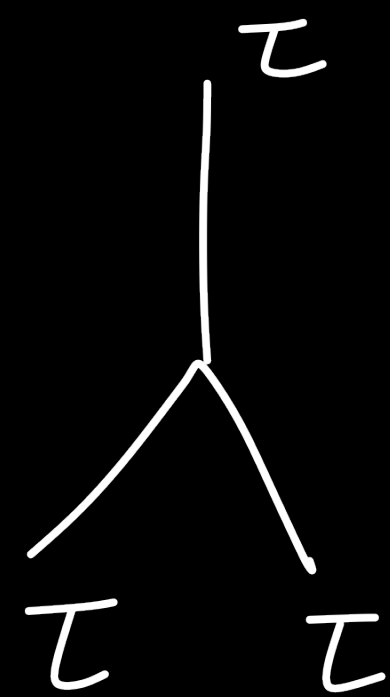
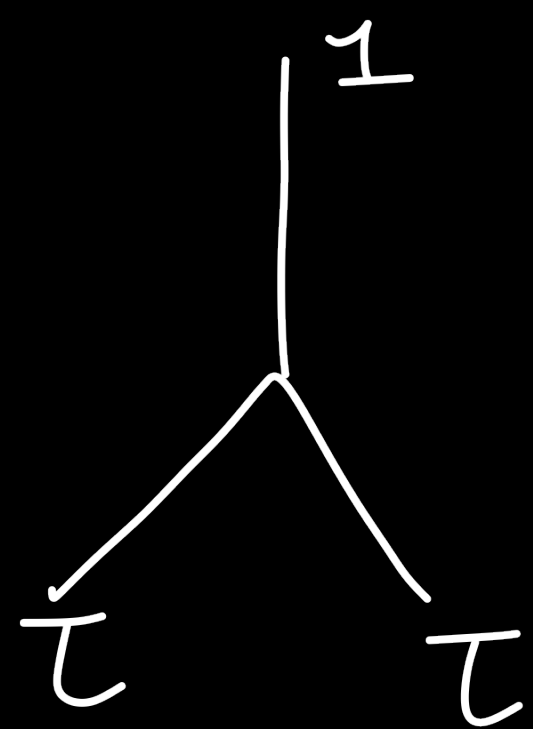
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Example

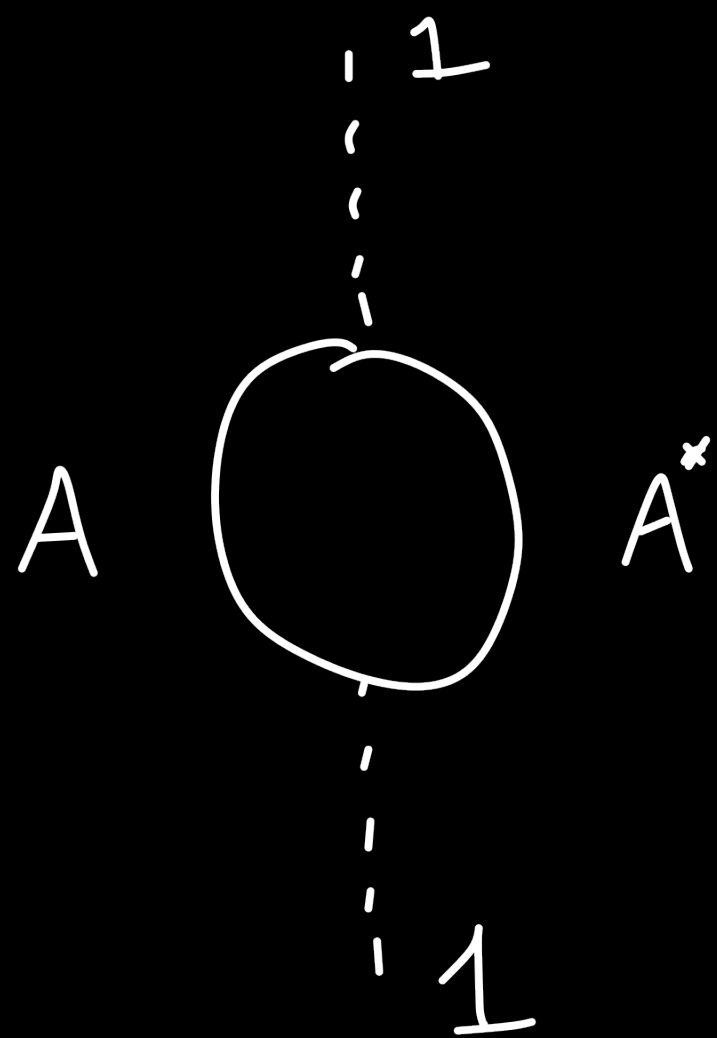
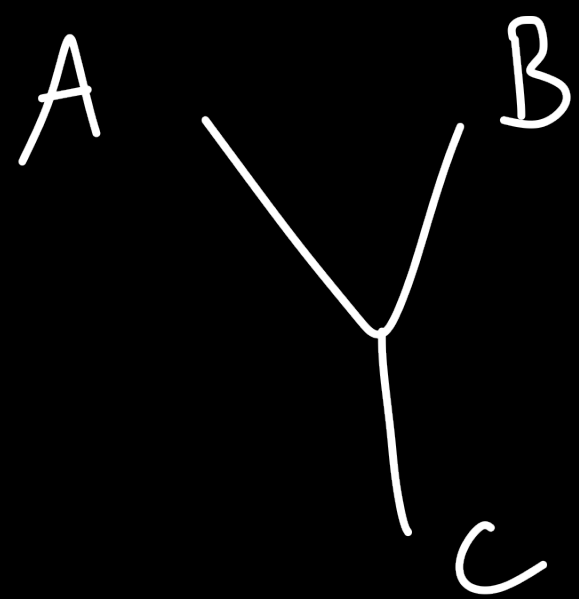
$$\text{Fib} \cong \{1, \tau\}.$$

$$\tau \otimes \tau = 1 \oplus \tau$$



Need some notion of conjugation

$\Rightarrow$  flip pictures upside down



$$\in \text{Hom}(1, 1) \cong \mathbb{C}$$

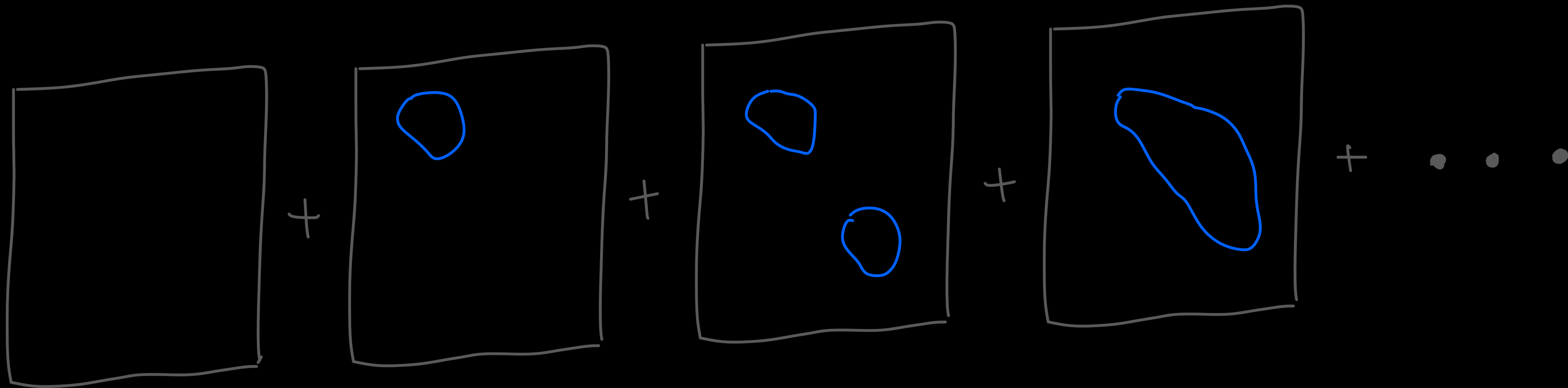
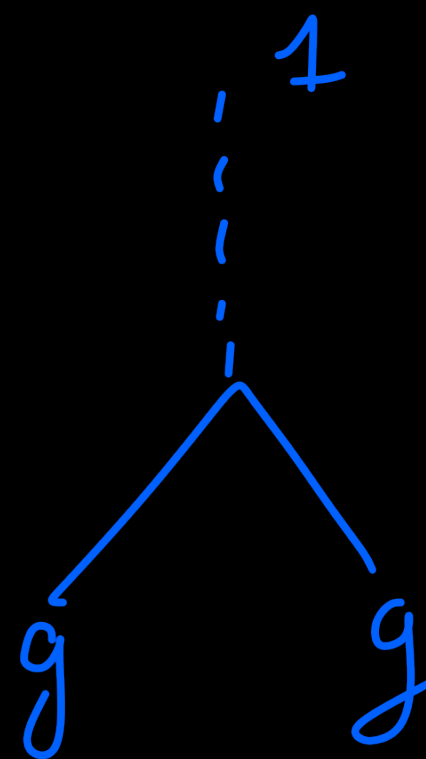
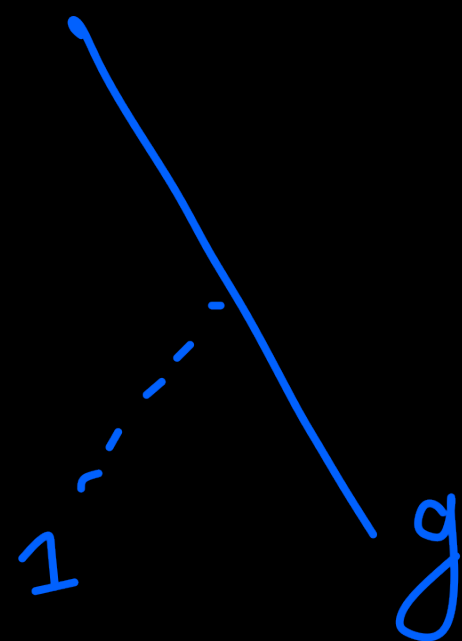
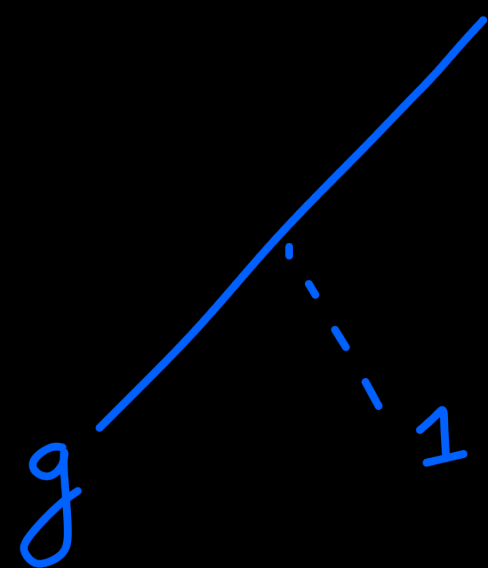
## String-net models

- 1) Draw a closed picture from  $\mathcal{L}$ .
- 2) Keep adding pictures obtained by local moves
- 3)  $\Rightarrow$  ground state



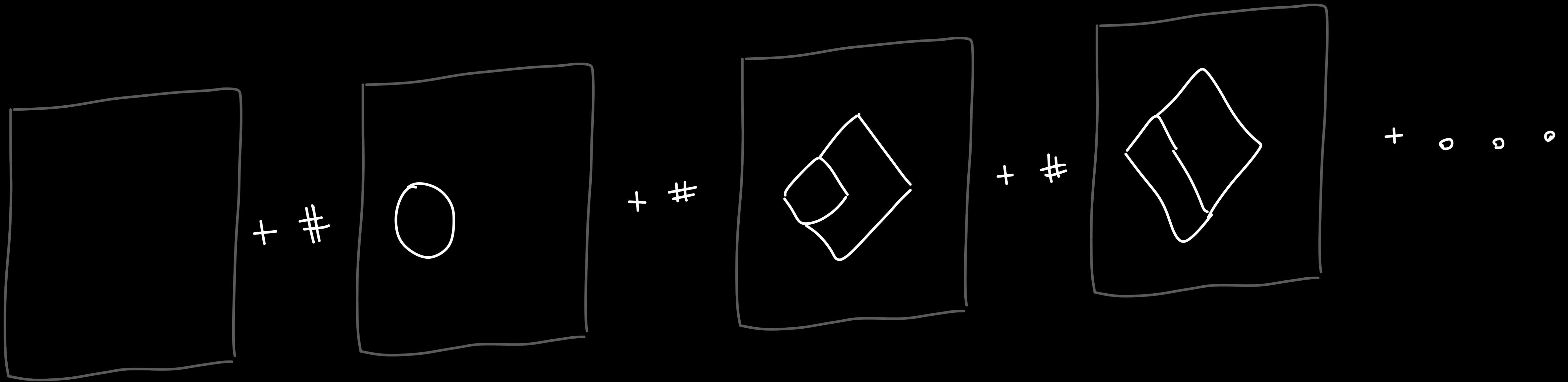
Example

$$\underline{\text{Toric code}} = \text{Vec } \mathbb{Z}/2 = \{1, g\}.$$



Example

# Double Fibonacci

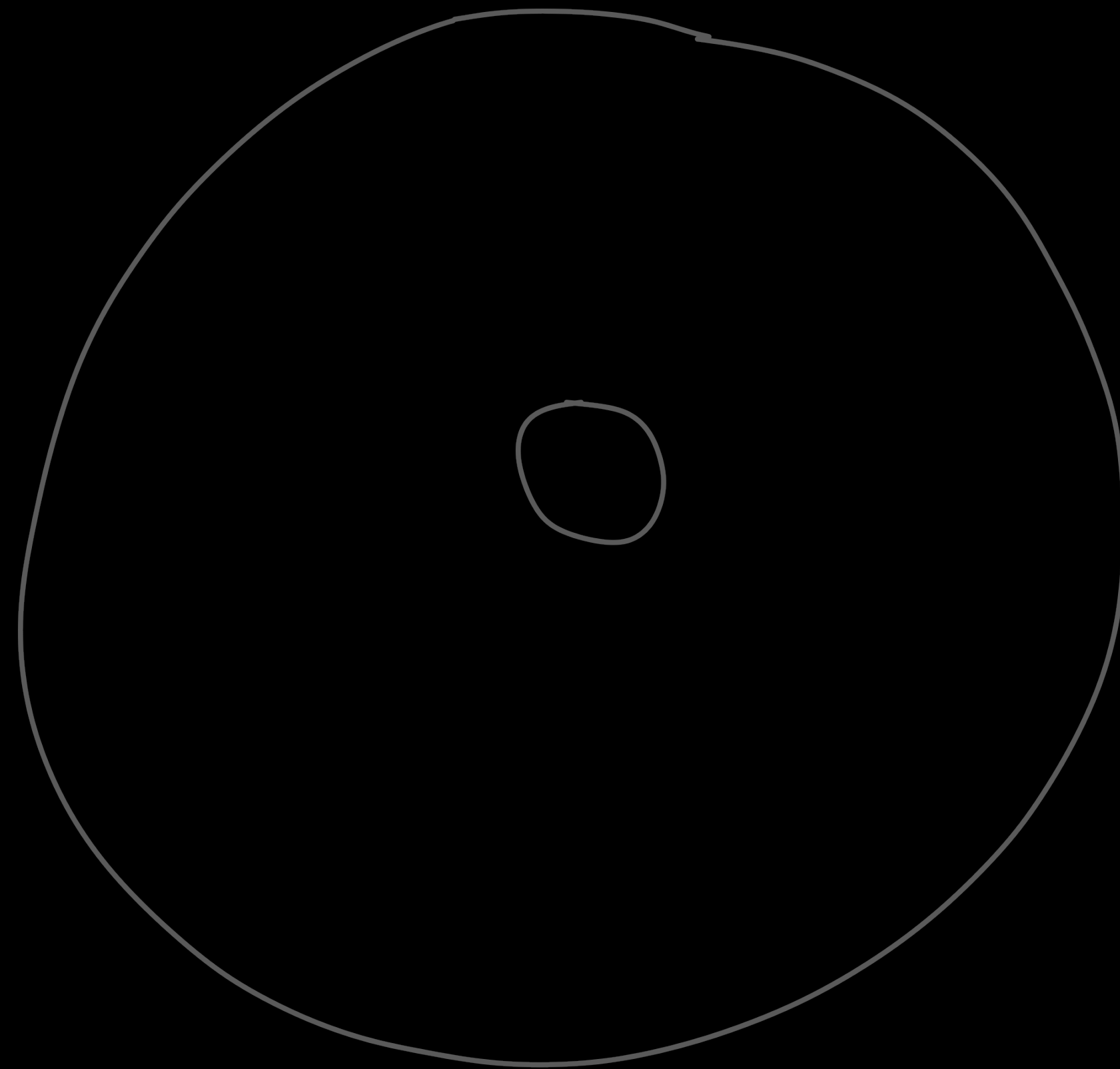


That's bare string nets

Questions?

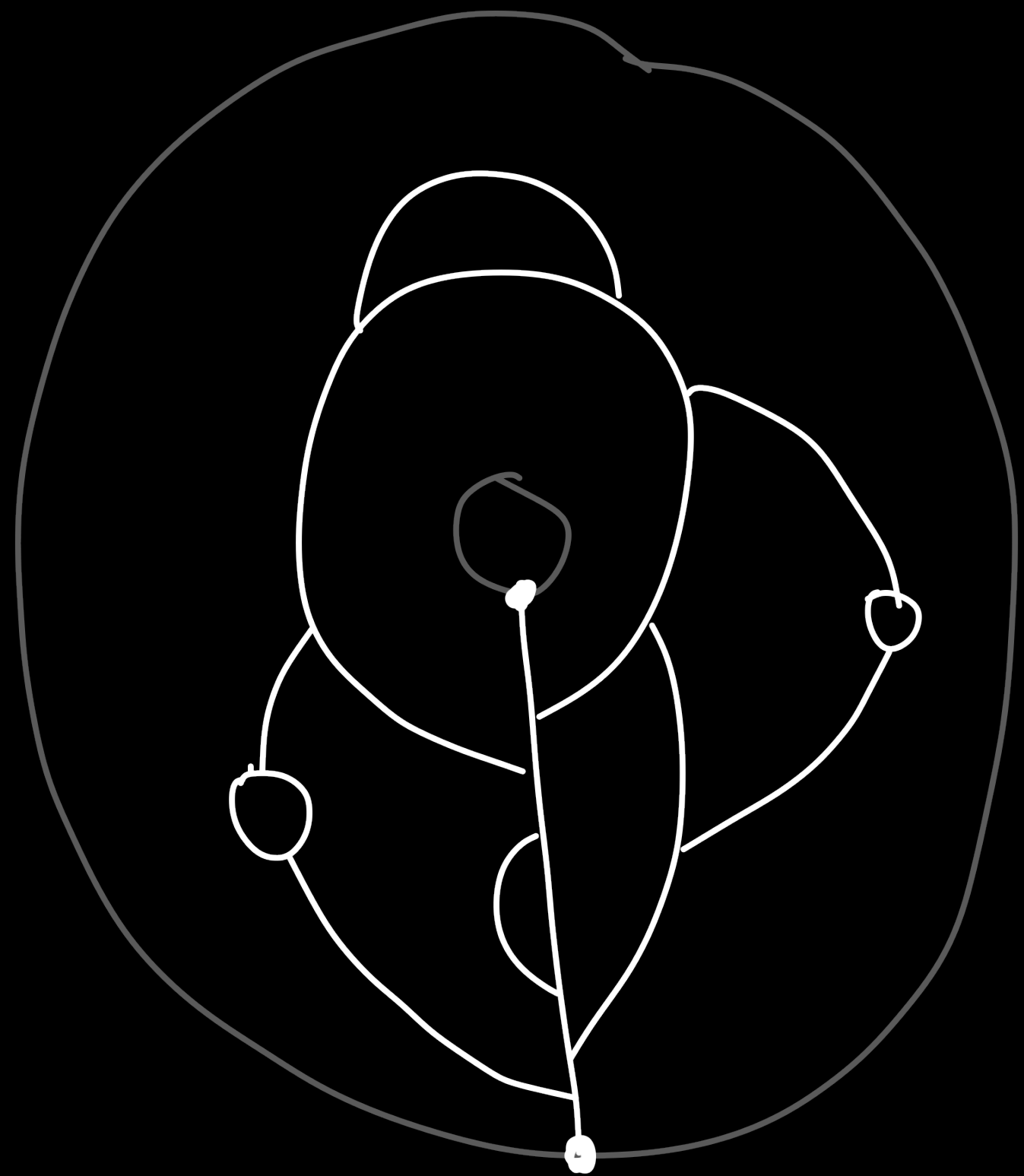
# Excitations

Modify the pictures at a point

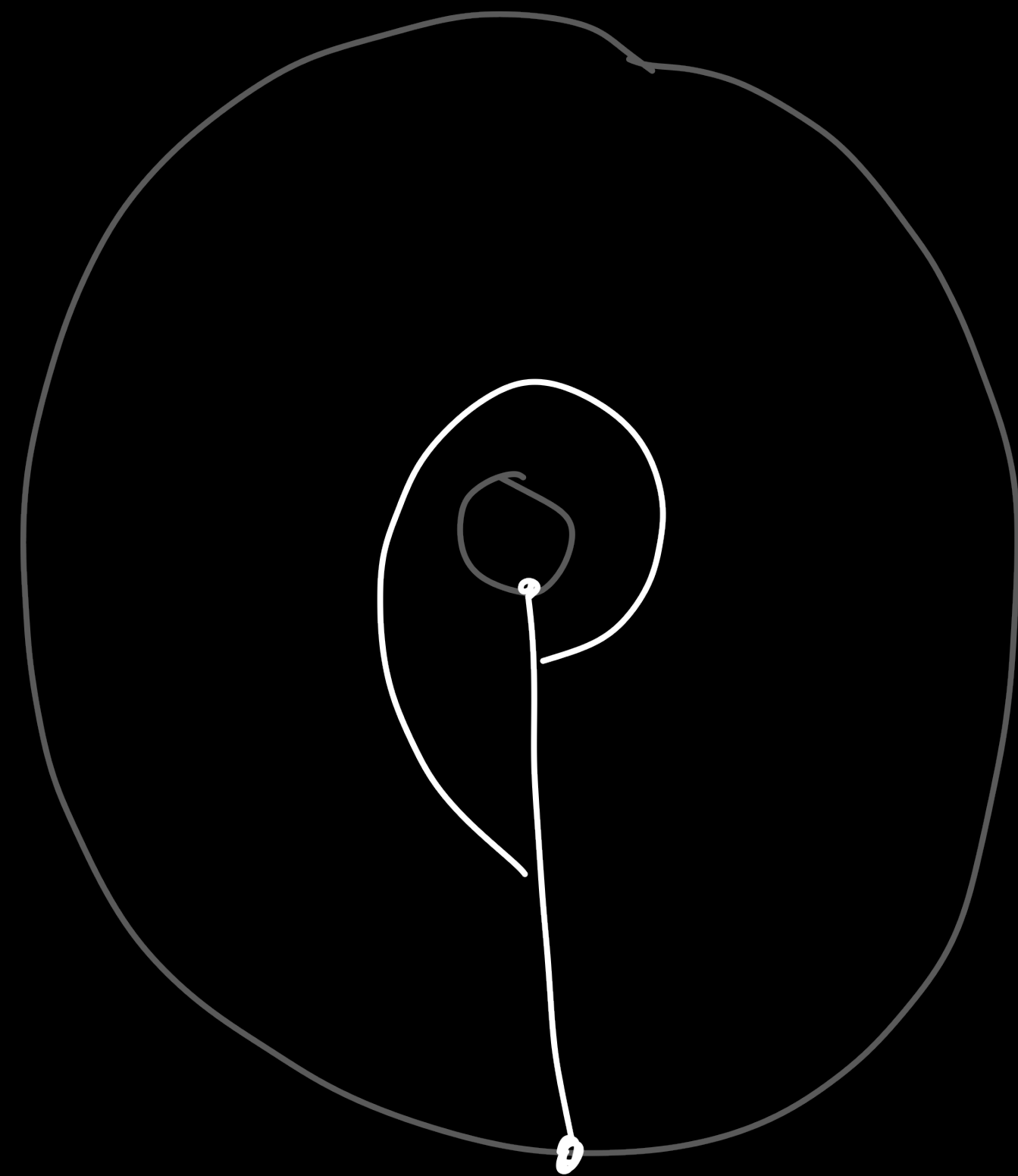


# Excitations

Modify the pictures at a point

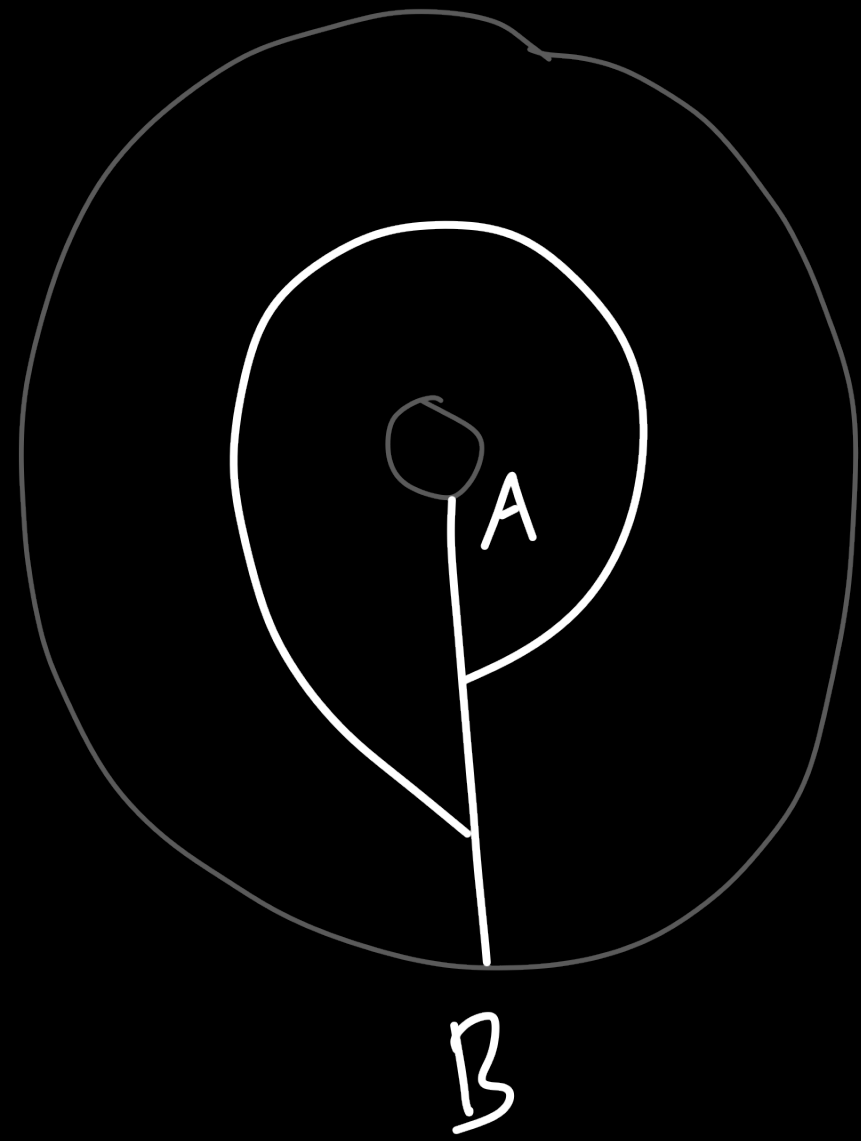


local moves



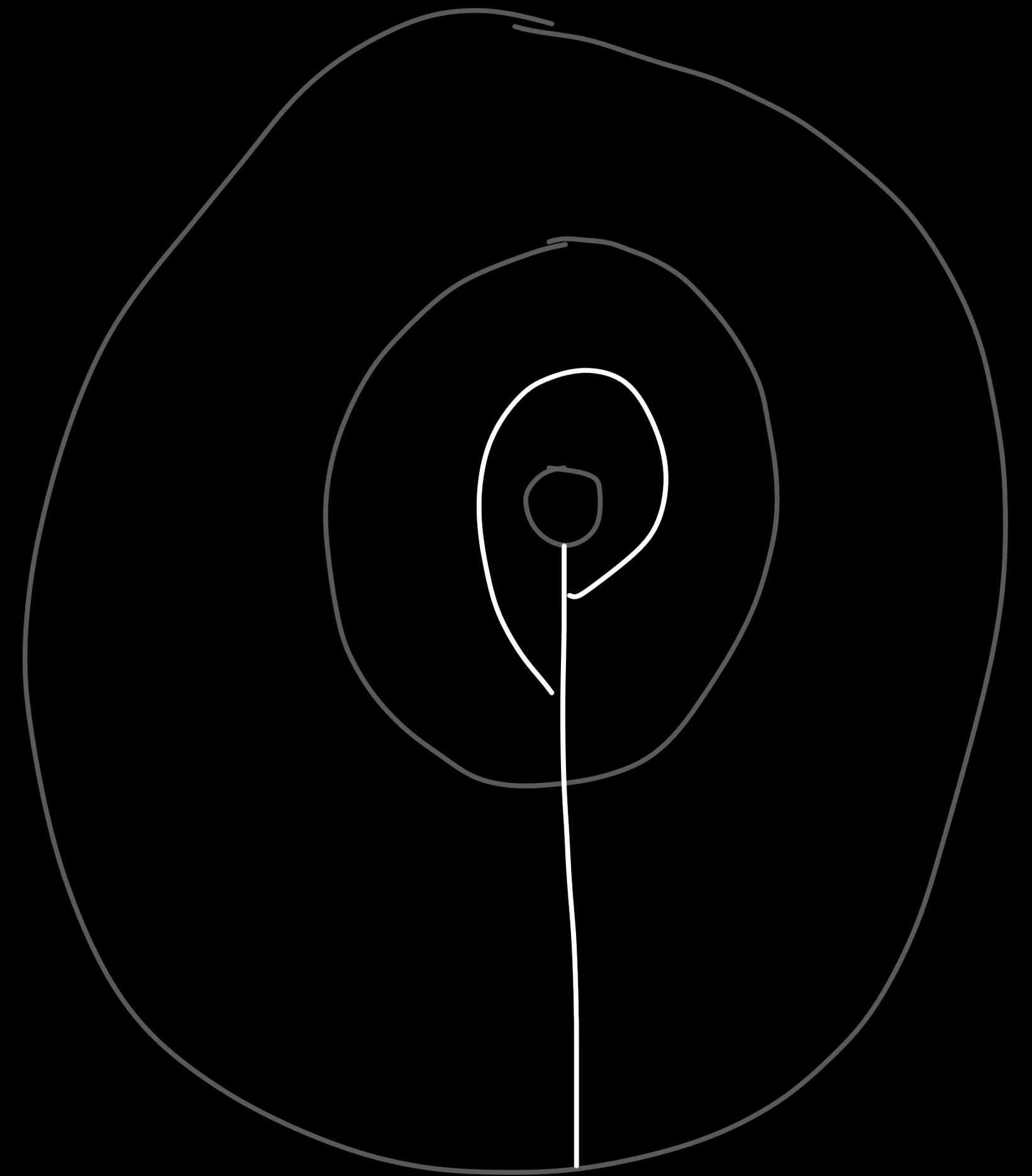
# Tube category

morphisms



$$: A \rightarrow B$$

composition

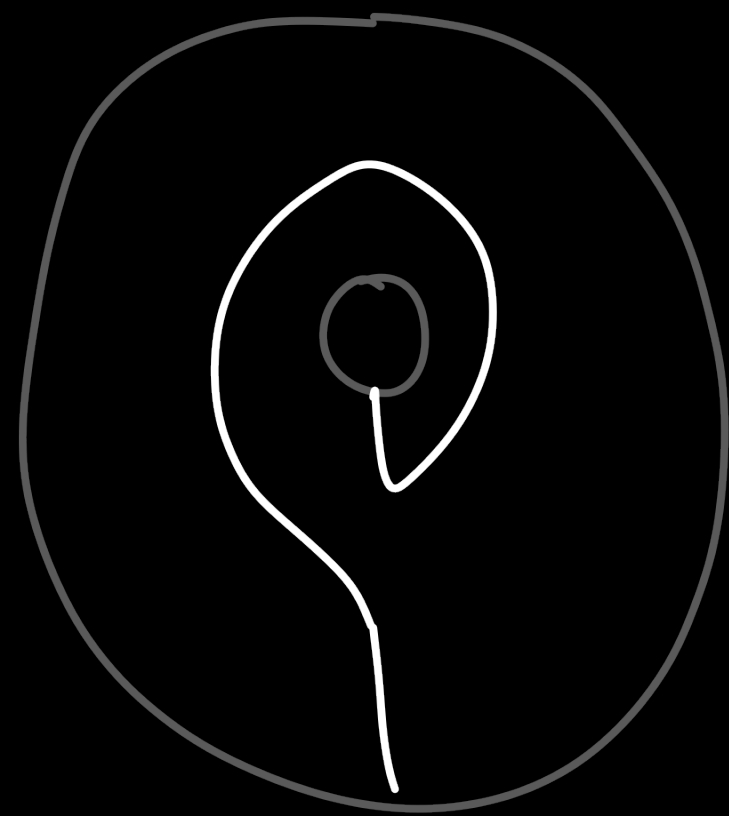
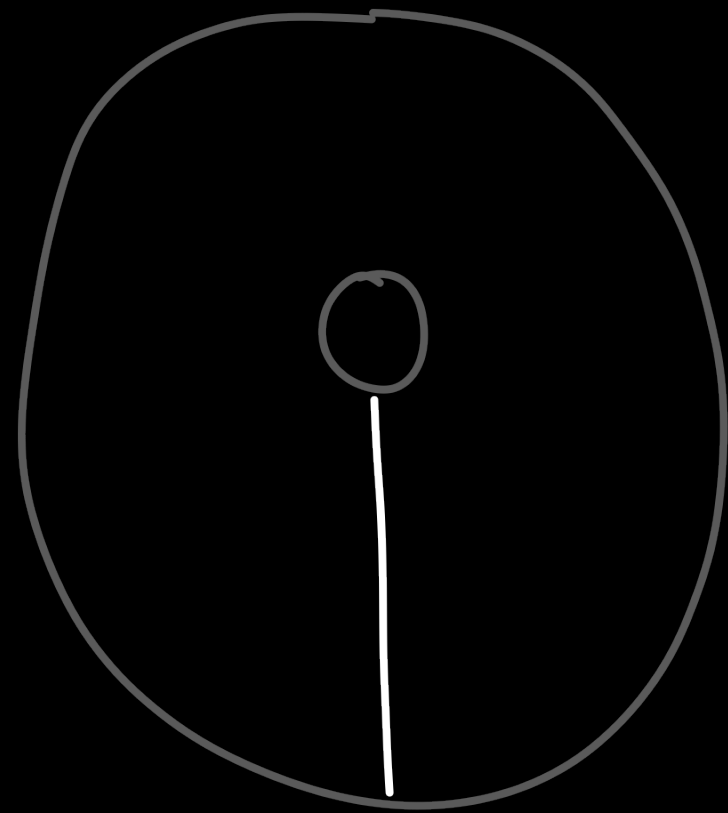
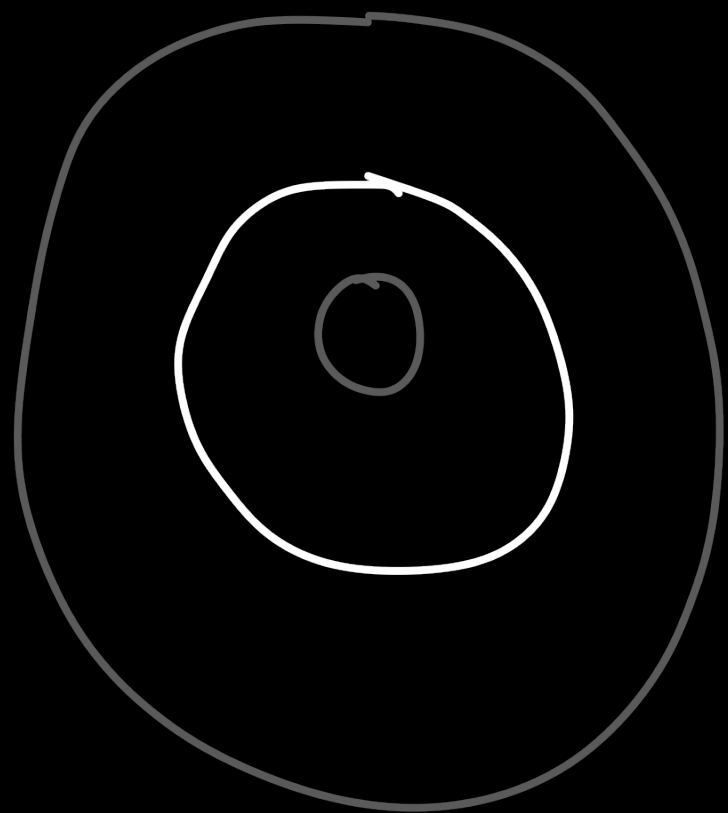
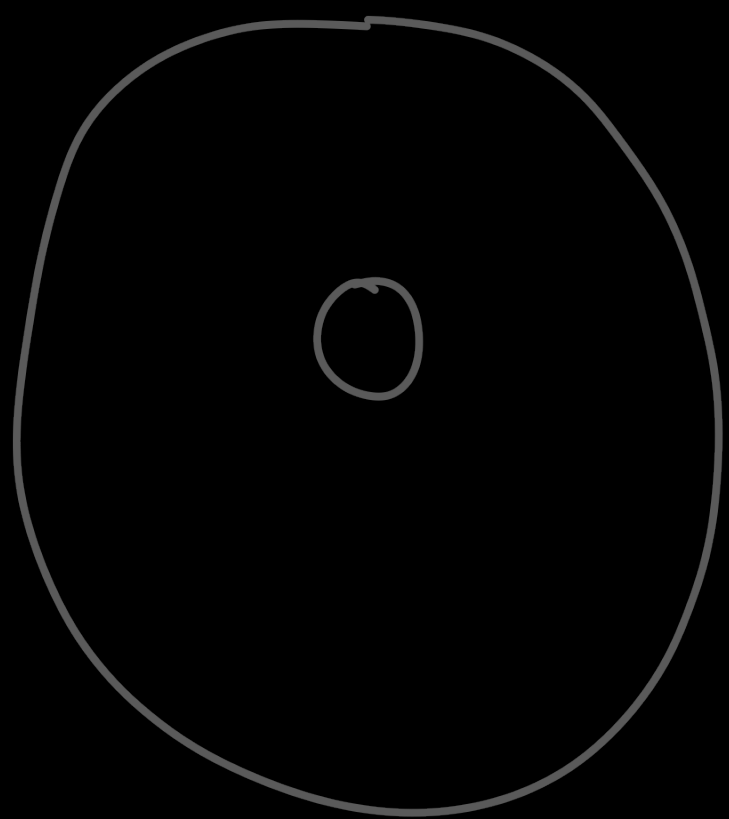
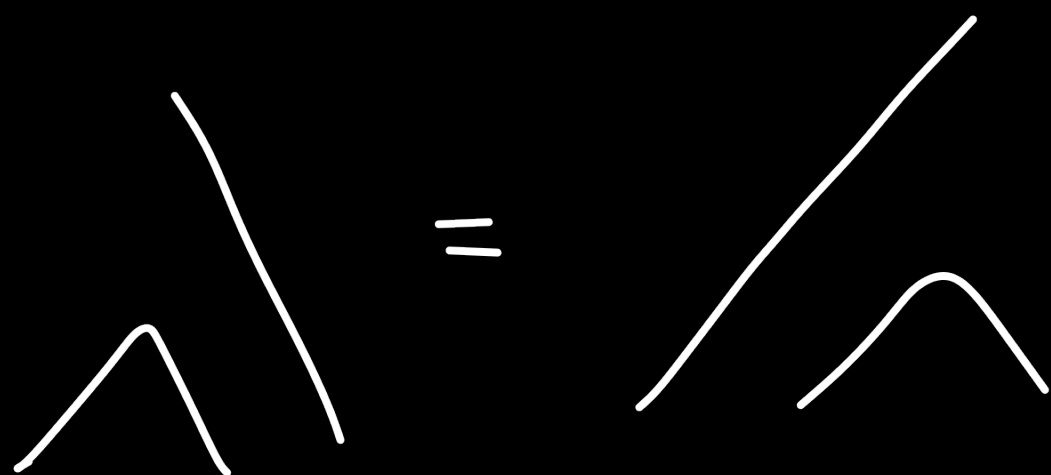


Excitations = Irreducible representations of  
tube category

Categorical representation:  $\begin{array}{ccc} \text{object } A & \longmapsto & \text{Vector space } V_A \\ (f: A \rightarrow B) & \longmapsto & (L_f: V_A \rightarrow V_B) \end{array}$

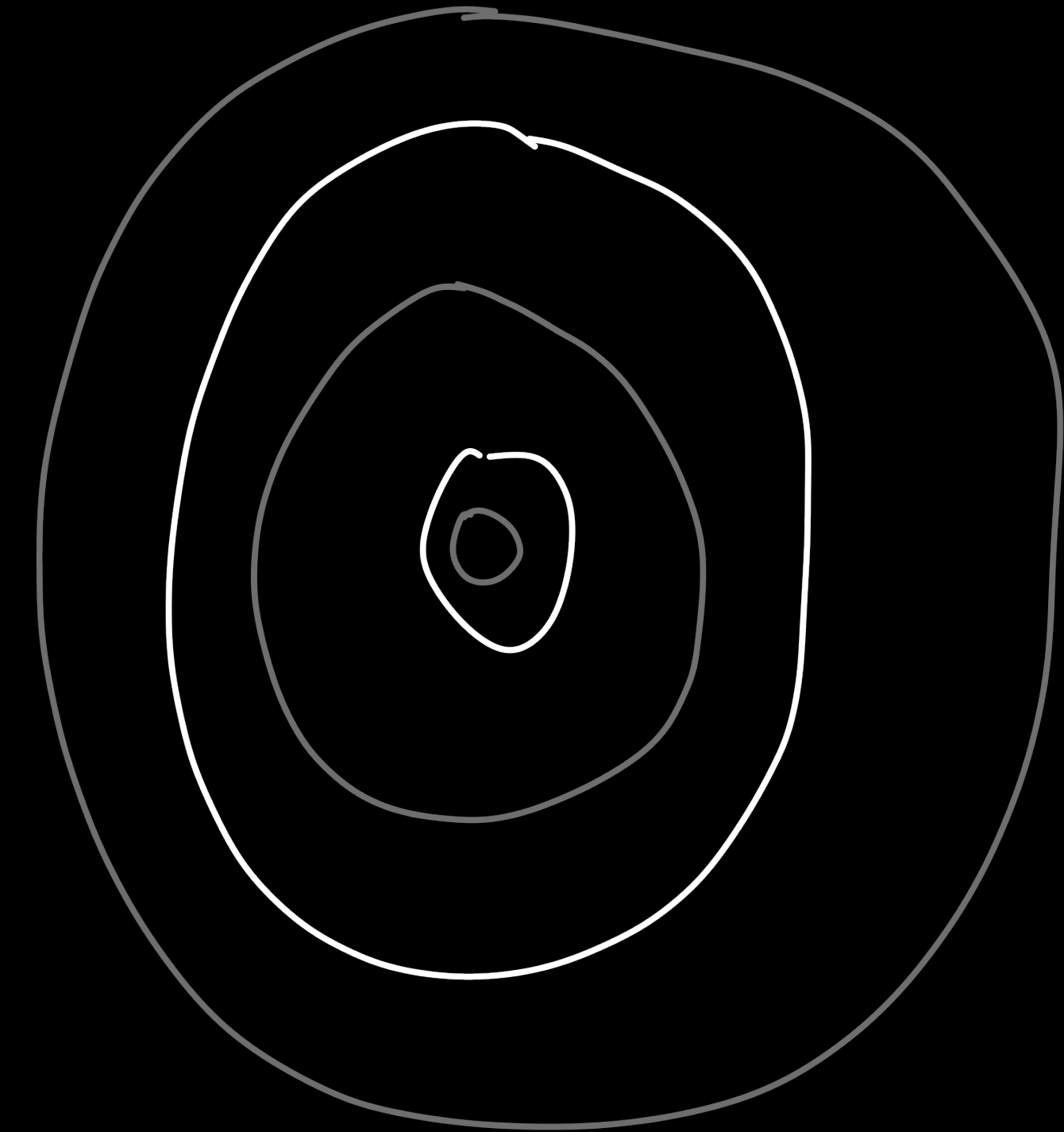
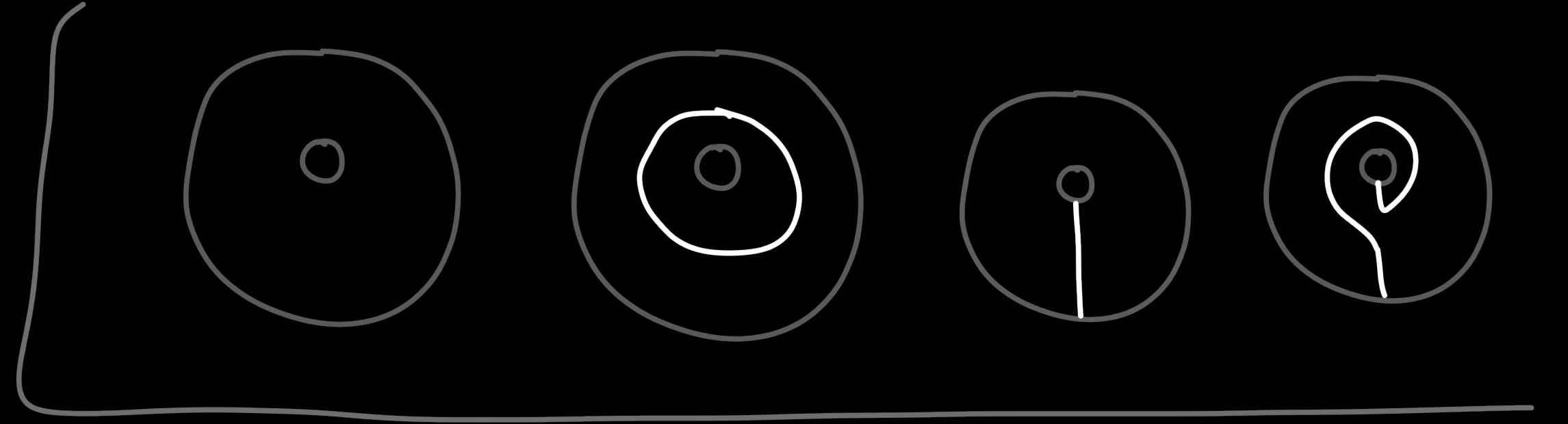
Example

Toric code  $\equiv$  Vec  $(\mathbb{Z}/2)$

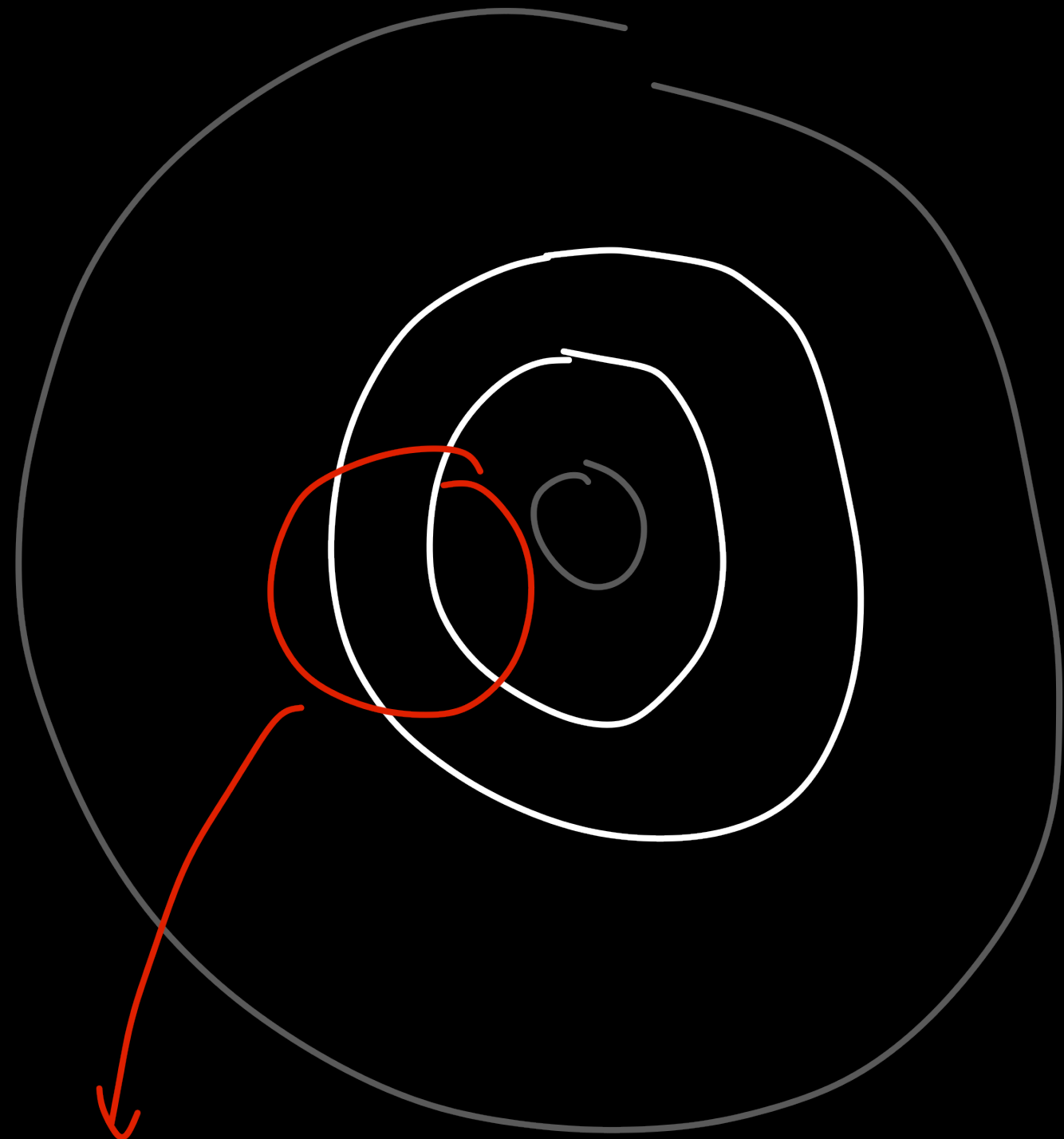




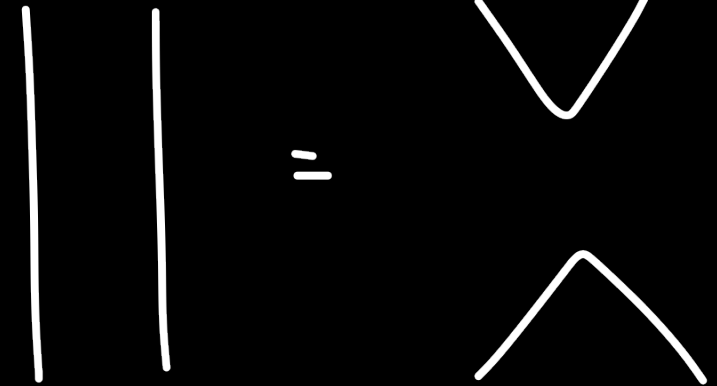
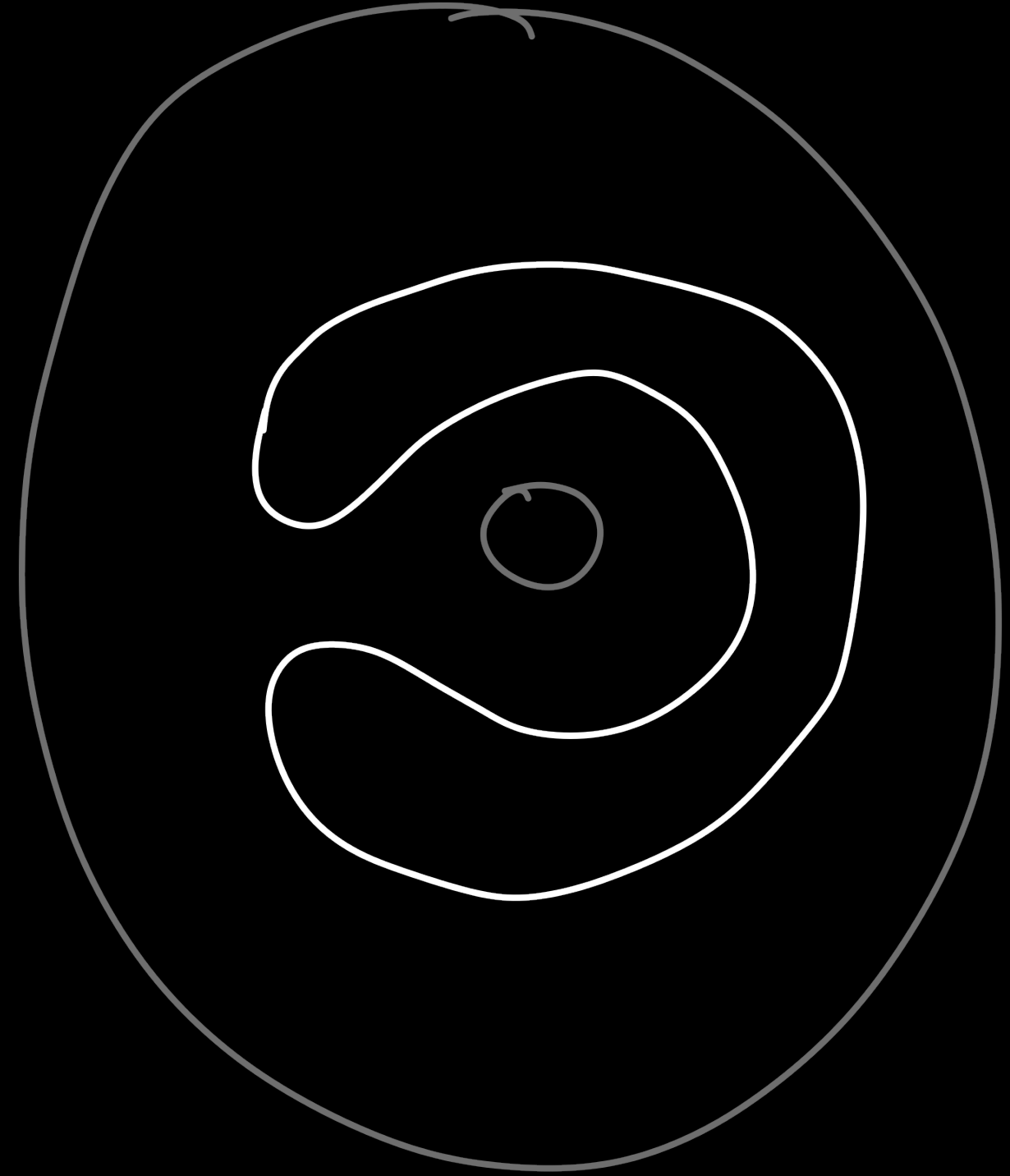
Example



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# Example

$$P_4 = \frac{1}{2} \left( \text{circle with } 0 \text{ inside} + \text{circle with } 0 \text{ inside and } 0 \text{ circled} \right)$$

$$P_e = \frac{1}{2} \left( \text{circle with } 0 \text{ inside} - \text{circle with } 0 \text{ inside and } 0 \text{ circled} \right)$$

$$P_m = \frac{1}{2} \left( \text{circle with } 0 \text{ inside and a vertical line below it} + \text{circle with } 0 \text{ inside and a vertical line below it and } 0 \text{ circled} \right)$$

$$P_{em} = \frac{1}{2} \left( \text{circle with } 0 \text{ inside and a vertical line below it} - \text{circle with } 0 \text{ inside and a vertical line below it and } 0 \text{ circled} \right)$$

$$\begin{array}{c} \bullet \\ | \\ e \end{array} \begin{array}{c} \bullet \\ | \\ m \end{array} = \frac{1}{2} \left( \text{Diagram 1} - \text{Diagram 2} \right) \times \frac{1}{2} \left( \text{Diagram 3} + \text{Diagram 4} \right)$$

Diagram 1: A large circle with a small circle inside.   
 Diagram 2: A large circle with a small circle inside, and a vertical line from the top of the small circle to the top of the large circle.   
 Diagram 3: A large circle with a vertical line from the top to the center.   
 Diagram 4: A large circle with a vertical line from the top to the center, and a small circle on the line.

$$= \frac{1}{4} \left( \text{Diagram 5} + \text{Diagram 6} - \text{Diagram 7} - \text{Diagram 8} \right)$$

$$= \frac{1}{4} \left( \text{Diagram 9} + \text{Diagram 10} - \text{Diagram 11} - \text{Diagram 12} \right)$$

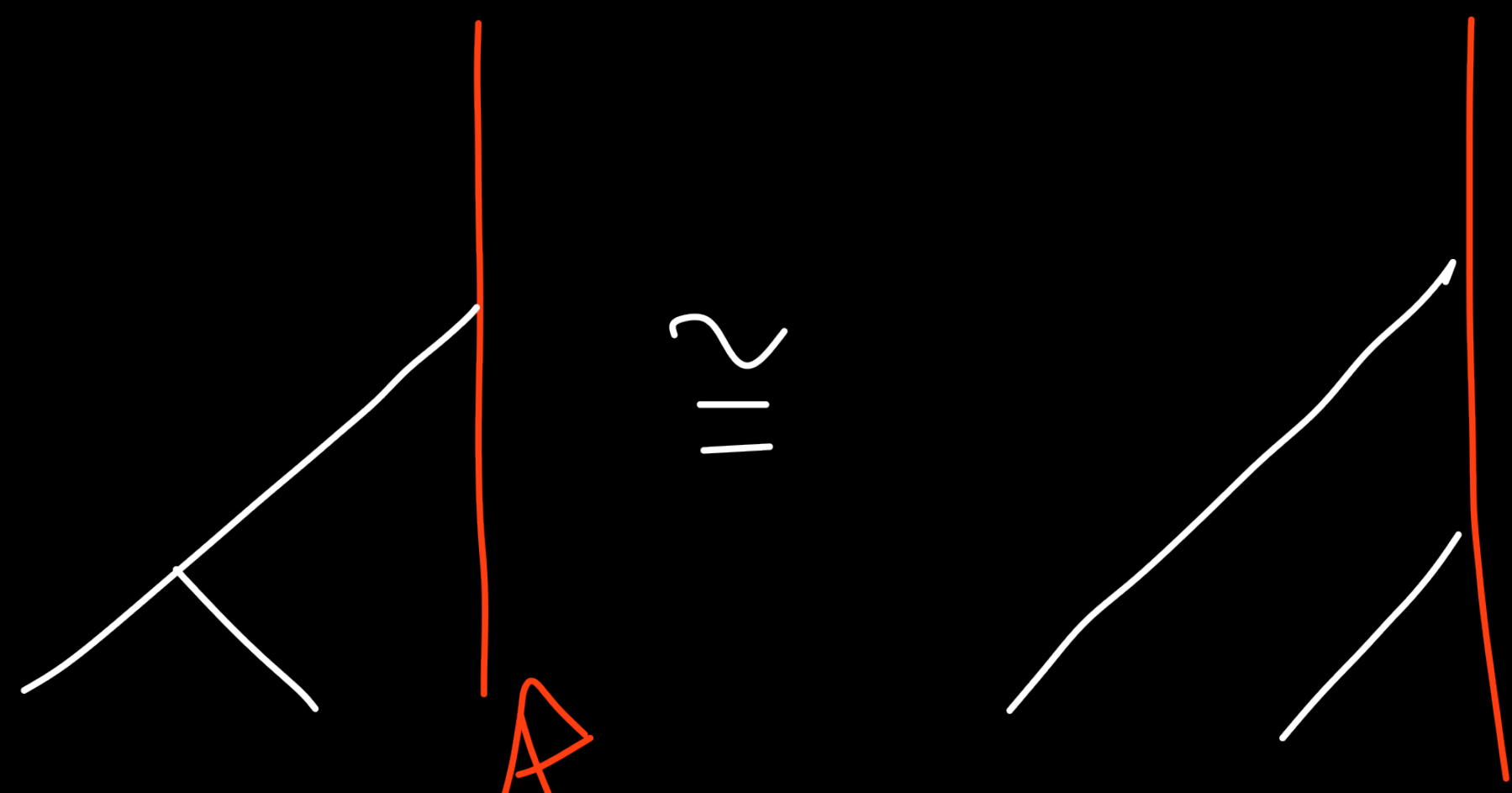
Diagram 5: Square with a small circle and a vertical line.   
 Diagram 6: Square with a small circle and a loop.   
 Diagram 7: Square with a small circle and a vertical line, and a larger circle to the left.   
 Diagram 8: Square with a small circle and a loop, and a larger circle to the left.   
 Diagram 9: Square with a loop and a vertical line.   
 Diagram 10: Square with a loop and a vertical line, and a larger circle to the left.   
 Diagram 11: Square with a loop and a vertical line, and a larger circle to the left, and a vertical line from the top of the loop.   
 Diagram 12: Square with a loop and a vertical line, and a larger circle to the left, and a vertical line from the top of the loop.

$$= - \frac{1}{2} \left( \text{Diagram 13} + \text{Diagram 14} - \text{Diagram 15} - \text{Diagram 16} \right) = - \begin{array}{c} \bullet \\ | \\ e \end{array} \begin{array}{c} \bullet \\ | \\ m \end{array}$$

Diagram 13: Square with a loop and a vertical line.   
 Diagram 14: Square with a loop and a vertical line, and a larger circle to the left.   
 Diagram 15: Square with a loop and a vertical line, and a larger circle to the left, and a vertical line from the top of the loop.   
 Diagram 16: Square with a loop and a vertical line, and a larger circle to the left, and a vertical line from the top of the loop.

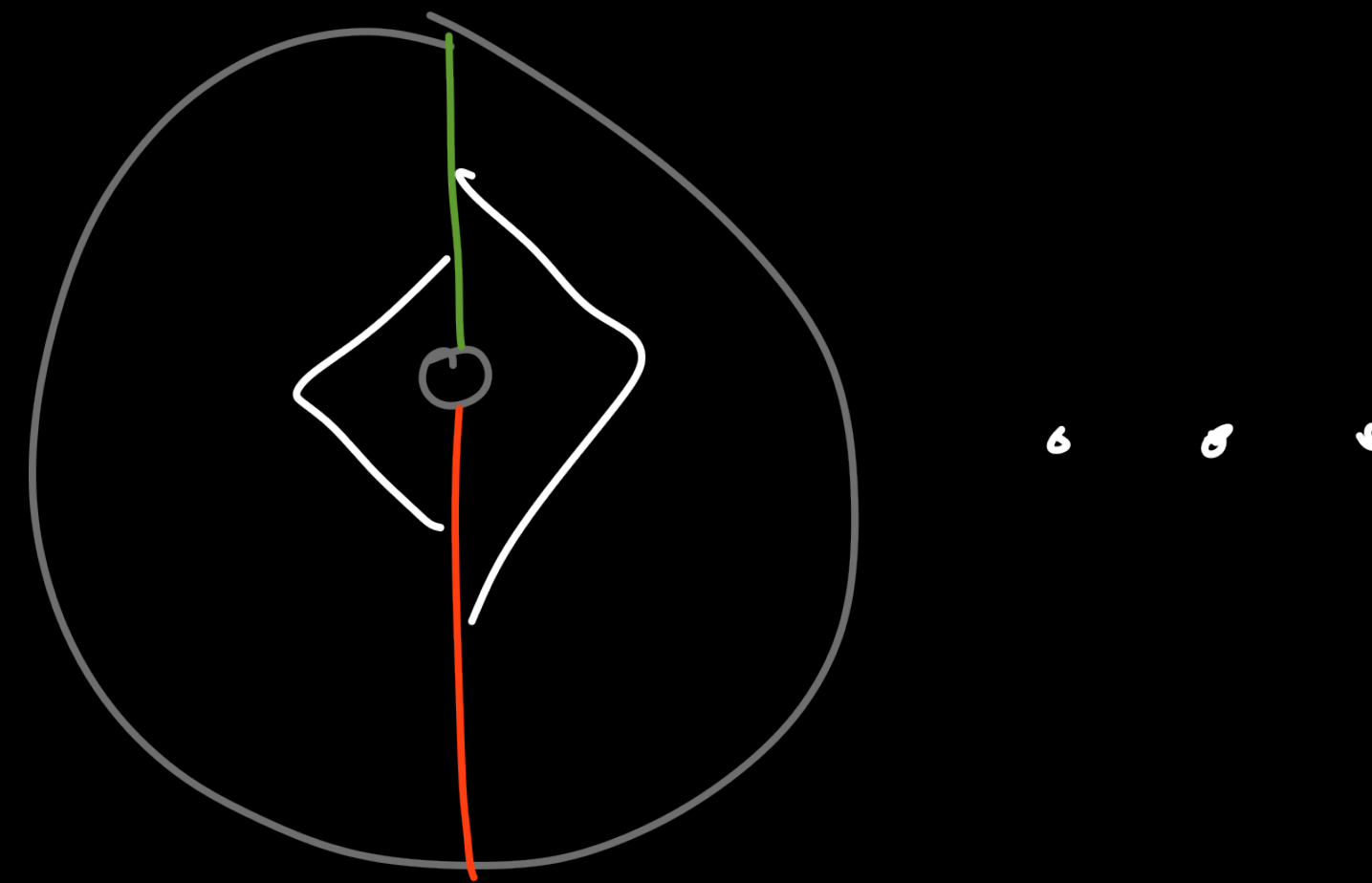
Can compute fusion in the same way

Boundaries: How can we modify a 1D region



More categories,  
this one a "module category"

Boundary excitations?  
combine these ideas



Questions?

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"Tensor categories: Etingof et al."